Finite dimensional rank 2 Nichols algebras of diagonal type II: Classification

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Abstract

The method of subquotients is developed and used to determine all finite dimensional rank 2 Nichols algebras of diagonal type over an arbitrary field of characteristic zero.

Key Words: Hopf algebra, Nichols algebra

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1 Introduction

This paper is the continuation of [5]. Let k be a field of characteristic zero, G an abelian group, $V \in^{kG}_{kG} \mathcal{YD}$ a Yetter–Drinfel'd module, and assume that the braiding of V is of diagonal type. Let $\mathcal{B}(V)$ denote the corresponding Nichols algebra. Our aim is to give an answer to the following question of Andruskiewitsch stated in his survey [1].

Question 5.40. Given a braided vector space V of diagonal type and dimension 2, decide when $\mathcal{B}(V)$ is finite dimensional. If so, compute $\dim \mathcal{B}(V)$, and give a "nice" presentation by generators and relations.

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In the previous part we proved finite dimensionality of several rank 2 Nichols algebras with help of Grañas differential operators [4], Kharchenkos theory [6] and full binary trees. Here it is shown that any finite dimensional rank 2 Nichols algebra of diagonal type is isomorphic to one of the examples in [5].

The starting point of our classification is the following observation. Under certain assumptions one can find elements of $\mathcal{B}(V)$ such that the subalgebra generated by them maps surjectively onto another Nichols algebra. The latter is called a subquotient of $\mathcal{B}(V)$. The existence of subquotients can be used in a very simple way to prove infinite dimensionality of Nichols algebras. For example, if there is a proper subquotient of $\mathcal{B}(V)$ which is isomorphic to $\mathcal{B}(V)$ or if the subquotient is infinite dimensional then $\mathcal{B}(V)$ is infinite dimensional.

The method of subquotients presented here is applicable also for Nichols algebras of higher rank. However our classification method is effective only together with the constructive part described in [5]. The latter uses full binary trees which have to be generalized or replaced if the rank of the Nichols algebra is greater than 2. It remains a challenging problem to find an appropriate structure which carries sufficiently many information needed to prove finiteness results and to read off the algebra structure of $\mathcal{B}(V)$.

The structure of the paper is as follows. First we formulate general conditions for the existence of subquotients. In order to allow an easier application some Corollaries are formulated. Then in Theorem 5 the list of all finite dimensional rank 2 Nichols algebras is given, this time sorted systematically by the entries of the braiding. The main part of this paper is devoted to the description of sufficiently many kinds of subquotients of $\mathcal{B}(V)$. In the last section the classification is splitted into six special cases. In each of them additional careful choices of subquotients are needed in order to obtain valuable criterions for the entries of the braiding.

We use the notation and conventions in [5], Section 2, which follow mainly [2].

2 Nichols algebras and subquotients

Suppose that k is a field of characteristic zero, G an abelian group, and $V \in {}^{kG}_{kG}\mathcal{YD}$ a finite dimensional Yetter–Drinfel'd module with completely reducible kG-action. Set $d := \dim_k V$. Let $\delta : V \to kG \otimes V$, $. : kG \otimes V \to V$, and $\sigma \in \mathrm{End}_k(V \otimes V)$ denote the left coaction of kG on V, the left action of kG on V, and the braiding of V, respectively. Let $\mathcal{B}(V)$ be the Nichols algebra generated by V. The aim of this paper is to determine all $V \in {}^{kG}_{kG}\mathcal{YD}$ such that $\dim_k V = 2$ and $\dim_k \mathcal{B}(V) < \infty$.

Let V^* denote the Yetter–Drinfel'd module (left) dual to V. Then the vector space $\mathcal{B}(V^*) \otimes kG$ together with the product

$$(f' \otimes g')(f'' \otimes g'') := f'(g'.f'') \otimes g'g'', \qquad f', f'' \in \mathcal{B}(V^*),\ g', g'' \in G,$$

and the coproduct

$$\Delta(f) := f \otimes 1 + \delta(f), \qquad \Delta(g) := g \otimes g, \qquad f \in V^*, g \in G,$$

is a Hopf algebra and will be denoted as usual by $\mathcal{B}(V^*)\#kG$. Recall the following lemma from [5].

Lemma 1. *There exists a unique action $\langle \cdot, \cdot \rangle$ of $\mathcal{B}(V^*)\#kG$ on $\mathcal{B}(V)$ satisfying*

$$\begin{aligned} \langle f, v\rangle = f(v), \quad \langle g, \rho\rangle = g.\rho \qquad & \textit{for } f \in V^*, v \in V, g \in G, \rho \in \mathcal{B}(V),\\ \langle f_1 f_2, \rho\rangle = \langle f_1, \langle f_2, \rho\rangle\rangle \qquad & \textit{for } f_1, f_2 \in \mathcal{B}(V^*)\#kG, \rho \in \mathcal{B}(V),\\ \langle f, \rho\rho'\rangle = \langle f_{(1)}, \rho\rangle\langle f_{(2)}, \rho'\rangle \qquad & \textit{for } f \in \mathcal{B}(V^*)\#kG, \rho, \rho' \in \mathcal{B}(V). \end{aligned}$$

Suppose that $d \in \mathbb{N}$, $g_i \in G$ for $1 \le i \le d$, $q_{ij} \in k \setminus \{0\}$ for $i, j \in \{1, 2, \ldots, d\}$, and $\{x_i \,|\, 1 \le i \le d\}$ is a basis of V such that

$$\delta(x_i) = g_i \otimes x_i, \qquad g_i.x_j = q_{ij}x_j.$$

Such a basis always exists and is called *a canonical basis* of V. The algebra $\mathcal{B}(V)$ admits an $\mathbb{N}_0^d$-grading such that $\deg x_i = \mathbf{e}_i$ where $\{\mathbf{e}_i \,|\, 1 \le i \le d\}$ is

a basis of the $\mathbb{N}_0$-module $\mathbb{N}_0^d$. The corresponding $\mathbb{Z}$-grading, the so called total grading, is defined by $\operatorname{totdeg} \rho := \sum_{i=1}^d n_i$ whenever $\rho \in \mathcal{B}(V)$, $\deg \rho = \sum_{i=1}^d n_i \mathbf{e}_i$. The $\mathbb{N}_0^d$-grading and the left kG-coaction on V induce a group homomorphism $g : \mathbb{Z}^d \to G$ and a bicharacter $\chi : \mathbb{Z}^d \times \mathbb{Z}^d \to k$ given by

$$g(\mathbf{e}_i) := g_i, \qquad \chi(\mathbf{e}_i, \mathbf{e}_j) := q_{ij}.$$

For notational convenience we will also write $g(x)$ and $\chi(x', x'')$ instead of $g(\deg x)$ and $\chi(\deg x', \deg x'')$ for homogeneous elements $x, x', x'' \in \mathcal{B}(V)$.

Let $\{y_i \,|\, 1 \le i \le d\}$ denote the dual basis of V^*. Then one gets

$$\delta(y_i) = g_i^{-1} \otimes y_i, \qquad g_i . y_j = q_{ij}^{-1} y_j, \qquad \sigma(y_i \otimes y_j) = q_{ij} y_j \otimes y_i. \tag{1}$$

Thus for diagonal braidings the linear map $\iota : V \to V^*$, $\iota(x_i) := y_i$ for $1 \le i \le d$, extends to an algebra isomorphism $\iota : \mathcal{B}(V) \to \mathcal{B}(V^*)$.

The corollaries of the following lemma are our main tools in the classification of Nichols algebras.

Lemma 2. *Let V and W be finite dimensional Yetter–Drinfel'd modules of diagonal type over an abelian group G and H, respectively. Set $d_V := \dim_k V$ and $d_W := \dim_k W$. Fix a canonical basis $\{w_i \,|\, 1 \le i \le d_W\}$ of W and elements $h_i \in H$, $1 \le i \le d_W$, such that $\delta(w_i) = h_i \otimes w_i$. Further, let $V_0 \subset \mathcal{B}(V)$ and $W_0 \subset \mathcal{B}(W)$ be Yetter–Drinfel'd submodules of dimension $n < \infty$. Let A and B denote the subalgebras of $\mathcal{B}(V)$ and $\mathcal{B}(W)$ generated by V_0 and W_0, respectively. Choose canonical bases $\{v_i^0 \,|\, 1 \le i \le n\}$ and $\{w_i^0 \,|\, 1 \le i \le n\}$ of V_0 and W_0, respectively. Suppose that*

- *$\langle \iota(w_i), B \rangle \subset B$ whenever $1 \le i \le d_W$,*
- *there exists a $\mathbb{Z}$-grading $\deg_0$ of $\mathcal{B}(V)$ such that the elements v_i^0 are homogeneous with respect to $\deg_0$ and $\deg_0(v_i^0) > 0$ for $1 \le i \le n$,*
- *there exist algebra automorphisms $\alpha_i \in \operatorname{Aut}_k(A)$, and k-linear maps $Y_i : A \to A$, $1 \le i \le d_W$, and $\varphi_0 : V_1 \to B$ where $V_1 := Y(V_0)$ and Y denotes the unital k-subalgebra of $\operatorname{End}_k(A)$ generated by $\{Y_i \,|\, 1 \le i \le d_W\}$, satisfying the following properties.*

- *For $1 \le i \le d_W$ there exist $n_i \in \mathbb{N}$ such that the maps α_i and Y_i are homogeneous of degree 0 and $-n_i$ with respect to $\deg_0$, respectively,*
- $Y_i(ab) = Y_i(a)b + \alpha_i(a)Y_i(b)$ *for all $a, b \in A$ and $1 \le i \le d_W$,*
- $\varphi_0(v_j^0) = w_j^0$, $\varphi_0(\alpha_i(v_j^0)) = h_i^{-1}.w_j^0$ *for $1 \le i \le d_W$ and $1 \le j \le n$,*
- $\varphi_0(Y_i(a)) = \langle \iota(w_i), \varphi_0(a)\rangle$ *for all $a \in V_1$ and $1 \le i \le d_W$.*

Then there exists a unique surjective k-algebra homomorphism $\varphi : A \to B$ such that $\varphi\restriction_{V_1} = \varphi_0$.

Proof. Since $\{v_i^0 \,|\, 1 \le i \le n\}$ and $\{w_i^0 \,|\, 1 \le i \le n\}$ generate A and B, respectively, and $\varphi(v_i^0) = w_i^0$, surjectivity and uniqueness of φ is clear. To prove existence one has to show that

$$a := \sum_{m\in\mathbb{N}_0, 1\le i_1,i_2,\dots,i_m\le n} \lambda_{m,i_1i_2\dots i_m} v_{i_1}^0 v_{i_2}^0 \cdots v_{i_m}^0 = 0, \quad \lambda_{m,i_1i_2\dots i_m} \in k \qquad (*)$$
$$\Rightarrow \sum_{m\in\mathbb{N}_0, 1\le i_1,i_2,\dots,i_m\le n} \lambda_{m,i_1i_2\dots i_m} w_{i_1}^0 w_{i_2}^0 \cdots w_{i_m}^0 = 0.$$

This can be done by induction over $\deg_0(a)$. Let $\mathcal{G}_m$ denote the set of homogeneous elements of A of degree m with respect to $\deg_0$. Since $\deg_0(v_i^0) > 0$ one has $\mathcal{G}_0 = k1$ and $\mathcal{G}_m = \{0\}$ for $m < 0$. Thus the assertion $(*)$ holds for $a \in \mathcal{G}_m$, $m \le 0$. Suppose now that $j \in \mathbb{N}_0$ and $(*)$ holds for all $a \in \mathcal{G}_m$, $m \le j$. Then for arbitrary $a \in \mathcal{G}_{j+1}$ we have to show that $\langle \iota(w_i), \tilde{a}\rangle = 0$ for $1 \le i \le d_W$ where $\tilde{a} := \sum_{m\in\mathbb{N}_0, 1\le i_1,i_2,\dots,i_m\le n} \lambda_{m,i_1i_2\dots i_m} w_{i_1}^0 w_{i_2}^0 \cdots w_{i_m}^0$. One computes

$$\begin{aligned}\langle \iota(w_i), &w_{i_1}^0 w_{i_2}^0 \cdots w_{i_m}^0\rangle = \sum_{l=0}^{m-1} (h_i^{-1}.w_{i_1}^0 \dots w_{i_l}^0)\langle \iota(w_i), w_{i_{l+1}}^0\rangle w_{i_{l+2}}^0 \cdots w_{i_m}^0 \\ &= \sum_{l=0}^{m-1} \varphi(\alpha_i(v_{i_1}^0 \dots v_{i_l}^0))\varphi_0(Y_i(v_{i_{l+1}}^0))\varphi(v_{i_{l+2}}^0 \dots v_{i_m}^0) = \varphi(Y_i(v_{i_1}^0 v_{i_2}^0 \cdots v_{i_m}^0)).\end{aligned}$$

Here the last two equations are valid because of the induction hypothesis and the assumption on the degrees of Y_i and α_i. Now $a = 0$ implies $Y_i(a) = 0$ for $1 \le i \le d_W$ and hence $\langle \iota(w_i), \tilde{a}\rangle = 0$. ∎

Corollary 3. *In the setting of Lemma 2 suppose additionally that $V = W$ and A is a proper subspace of B. Then B and hence $\mathcal{B}(V)$ are infinite dimensional vector spaces.*

Let $\mathcal{B}(V)^+$ denote the unique maximal ideal of $\mathcal{B}(V)$.

Corollary 4. *Let V be a Yetter–Drinfel'd module of diagonal type and $W \subset \mathcal{B}(V)^+$ an n-dimensional Yetter–Drinfel'd submodule where $n \in \mathbb{N}_0$. Choose a canonical basis $\{w_i \,|\, 1 \le i \le n\}$ of W and let A denote the subalgebra of $\mathcal{B}(V)$ generated by W. If there exist $\lambda_i \in k \setminus \{0\}$, $1 \le i \le n$, such that the restrictions of $\langle \iota(w_i), \cdot\rangle$ onto A are skew-primitive and $\langle \iota(w_i), w_j\rangle = \delta_{ij}\lambda_i$ then $\mathcal{B}(W)$ is a quotient algebra of A. In particular, $\dim_k \mathcal{B}(W) = \infty$ implies that $\dim_k \mathcal{B}(V) = \infty$.*

Definition 1. In the situation of Corollary 4 we say that $\mathcal{B}(W)$ *is a subquotient of* $\mathcal{B}(V)$ and write $\mathcal{B}(W) \trianglelefteq \mathcal{B}(V)$. If further $W \neq \emptyset$ and $W \neq V$ then we say that $\mathcal{B}(W)$ *is a proper subquotient of* $\mathcal{B}(V)$ and write $\mathcal{B}(W) \triangleleft \mathcal{B}(V)$.

Proof of Corollary 4. In Lemma 2 set $V_0 = W_0 := W$ and $v_i^0 := w_i^0 := w_i$ for $1 \le i \le n$. For the grading $\deg_0$ take the grading totdeg on $\mathcal{B}(V)$. Further, set $\alpha_i := h_i^{-1}.(\cdot)$ and $Y_i := \lambda_i^{-1}\langle \iota(w_i), \cdot\rangle$. Then Lemma 2 gives a surjective algebra homomorphism $\varphi : A \to \mathcal{B}(W)$ extending the identity $\varphi_0 = \mathrm{id} : W \to W$. ■

3 The main result

From now on consider only the case where $\dim_k V = 2$. For $n \in \mathbb{N}$, $n \ge 2$ let R_n denote the subset of k consisting of the primitive n^{th} roots of unity. The following theorem is the main result of this paper.

Theorem 5. *Let k be a field of characteristic zero and G an abelian group. Let $V \in {}^{kG}_{kG}\mathcal{YD}$ be a Yetter–Drinfel'd module with $\dim_k V = 2$ and completely reducible kG-action. Assume that the Nichols algebra $\mathcal{B}(V)$ is finite dimensional. Then there exists a canonical basis B of V such that the*

entries of the matrix $(q_{ij})_{i,j=1,2}$ *of the braiding of* V *with respect to* B *satisfy*

$$q_{12}q_{21} = 1 \text{ or } q_{12}q_{21}q_{22} = 1 \text{ or } q_{22} = -1 \text{ or } q_{22} \in R_3.$$

More precisely, B *can be chosen such that one of the following holds.*

1. $q_{12}q_{21} = 1$ *and* $q_{11}, q_{22} \in \cup_{n=2}^{\infty} R_n$.
2. *If* $q_{12}q_{21} \neq 1$, $q_{12}q_{21}q_{22} = 1$*:*
   - $q_{11}q_{12}q_{21} = 1$, $q_{12}q_{21} \in \cup_{n=2}^{\infty} R_n$.
   - $q_{11} = -1$, $q_{12}q_{21} \in \cup_{n=3}^{\infty} R_n$.
   - $q_{11} \in R_3$, $q_{12}q_{21} \in \cup_{n=2}^{\infty} R_n$, $q_{11}q_{12}q_{21} \neq 1$.
   - $q_{11} \in \cup_{n=4}^{\infty} R_n$, $q_{12}q_{21} \in \{q_{11}^{-2}, q_{11}^{-3}\}$.
   - $q_{12}q_{21} \in R_8$, $q_{11} = (q_{12}q_{21})^2$.
   - $q_{12}q_{21} \in R_{24}$, $q_{11} = (q_{12}q_{21})^6$.
   - $q_{12}q_{21} \in R_{30}$, $q_{11} = (q_{12}q_{21})^{12}$.
3. *If* $q_{12}q_{21} \neq 1$, $q_{11}q_{12}q_{21} \neq 1$, $q_{12}q_{21}q_{22} \neq 1$, $q_{22} = -1$, $q_{11} \in R_2 \cup R_3$*:*
   - $q_{11} = -1$, $q_{12}q_{21} \in \cup_{n=3}^{\infty} R_n$.
   - $q_{11} \in R_3$, $q_{12}q_{21} \in \{q_{11}, -q_{11}\}$.
   - $q_0 := q_{11}q_{12}q_{21} \in R_{12}$, $q_{11} = q_0^4$.
   - $q_{12}q_{21} \in R_{12}$, $q_{11} = -(q_{12}q_{21})^2$.
   - $q_{12}q_{21} \in R_9$, $q_{11} = (q_{12}q_{21})^{-3}$.
   - $q_{12}q_{21} \in R_{24}$, $q_{11} = -(q_{12}q_{21})^4$.
   - $q_{12}q_{21} \in R_{30}$, $q_{11} = -(q_{12}q_{21})^5$.
4. *If* $q_{12}q_{21} \neq 1$, $q_{11}q_{12}q_{21} \neq 1$, $q_{12}q_{21}q_{22} \neq 1$, $q_{22} = -1$, $q_{11} \notin R_2 \cup R_3$*:*
   - $q_{11} \in \cup_{n=5}^{\infty} R_n$, $q_{12}q_{21} = q_{11}^{-2}$.
   - $q_{11} \in R_5 \cup R_8 \cup R_{12} \cup R_{14} \cup R_{20}$, $q_{12}q_{21} = q_{11}^{-3}$.

- $q_{11} \in R_{10} \cup R_{18},\ q_{12}q_{21} = q_{11}^{-4}$.
- $q_{11} \in R_{14} \cup R_{24},\ q_{12}q_{21} = q_{11}^{-5}$.
- $q_{12}q_{21} \in R_8,\ q_{11} = (q_{12}q_{21})^{-2}$.
- $q_{12}q_{21} \in R_{12},\ q_{11} = (q_{12}q_{21})^{-3}$.
- $q_{12}q_{21} \in R_{20},\ q_{11} = (q_{12}q_{21})^{-4}$.
- $q_{12}q_{21} \in R_{30},\ q_{11} = (q_{12}q_{21})^{-6}$.

5. *If* $q_{12}q_{21} \neq 1,\ q_{11}q_{12}q_{21} \neq 1,\ q_{12}q_{21}q_{22} \neq 1,\ q_{11} \neq -1,\ q_{22} \in R_3$*:*

- $q_0 := q_{11}q_{12}q_{21} \in R_{12},\ q_{11} = q_0^4,\ q_{22} = -q_0^2$.
- $q_{12}q_{21} \in R_{12},\ q_{11} = q_{22} = -(q_{12}q_{21})^2$.
- $q_{12}q_{21} \in R_{24},\ q_{11} = (q_{12}q_{21})^{-6},\ q_{22} = (q_{12}q_{21})^{-8}$.
- $q_{11} \in R_{18},\ q_{12}q_{21} = q_{11}^{-2},\ q_{22} = -q_{11}^3$.
- $q_{11} \in R_{30},\ q_{12}q_{21} = q_{11}^{-3},\ q_{22} = -q_{11}^5$.

Remark. The list of Nichols algebras given in Theorem 5 coincides with the one of [5, Theorem 4]. Here the examples are sorted systematically by their structure constants whereas in [5, Theorem 4] they are listed by their tree types. Therefore the Nichols algebras given in Theorem 5 are precisely all finite dimensional rank two Nichols algebras of diagonal type over the field k. ■

4 Construction of subquotients

As in Section 2 let $\{x_1, x_2\}$ denote a canonical basis of V. In this paper we want to classify all Nichols algebras $\mathcal{B}(V)$ satisfying

$$\dim_k \mathcal{B}(V) < \infty. \tag{A0}$$

To do so we determine the necessary additional conditions for the structure constants q_{ij}, $i, j \in \{1,2\}$.

4.1 The results in this subsection essentially coincide with the assertion of Lemma 3.7 in [3].

Note that $\langle y_i, x_i^m \rangle = \sum_{j=0}^{m-1} q_{ii}^{-j} x_i^{m-1}$ and the characteristic of k is zero. Thus assumption (A0) implies that

$$q_{11}, q_{22} \in \bigcup_{n=2}^{\infty} R_n. \tag{A1}$$

Set $x_{21} := x_1 x_2 - q_{12} x_2 x_1$ and $y_{21} := \iota(x_{21})$. Then by Equation (1) and Lemma 1 one gets

$$\langle y_1, x_{21} \rangle = 0, \quad \langle y_2, x_{21} \rangle = (q_{21}^{-1} - q_{12}) x_1, \quad \langle y_{21}, x_{21} \rangle = q_{21}^{-1} - q_{12},$$
$$\Delta(y_{21}) = y_{21} \otimes 1 + g_1^{-1} g_2^{-1} \otimes y_{21} + (1 - q_{12} q_{21}) y_1 g_2^{-1} \otimes y_2.$$

Now if (A0) holds then $x_{21}^m = 0$ for some $m \in \mathbb{N}$. On the other hand

$$\langle y_{21}, x_{21}^m \rangle = \langle y_{21}, x_{21} \rangle \sum_{j=0}^{m-1} (q_{11} q_{12} q_{21} q_{22})^{-j} x_{21}^{m-1}.$$

Thus (A0) implies that

$$q_{12} q_{21} = 1 \text{ or } q_{11} q_{12} q_{21} q_{22} \in \bigcup_{n=2}^{\infty} R_n. \tag{A2}$$

Let $z_0 := x_2$, $z_{i+1} := x_1 z_i - q_{11}^i q_{12} z_i x_1$, and $\hat{z}_i := \iota(z_i)$ for $i \in \mathbb{N}_0$. Set $b_0 := 1$, $b_i := \prod_{j=0}^{i-1} (1 - q_{11}^j q_{12} q_{21})$ for $i \in \mathbb{N}$, $(0)_p := 0$, $(i)_p := \sum_{j=0}^{i-1} p^j$, $(0)_p^! := 1$, $(i)_p^! := (1)_p (2)_p \cdots (i)_p$ for $i \in \mathbb{N}$ and $p \in k$, and $\binom{i}{j}_p := (i)_p^! / ((j)_p^! (i-j)_p^!)$ for $i, j \in \mathbb{N}_0$, $j \leq i$. Then one can prove by induction over i that

$$\langle y_1, z_i \rangle = 0, \quad \langle y_2, z_i \rangle = q_{21}^{-i} b_i x_1^i, \quad \langle \hat{z}_j, z_i \rangle = \langle y_1^j y_2, z_i \rangle = \frac{q_{21}^{-i} b_i (i)_{q_{11}^{-1}}^!}{(i-j)_{q_{11}^{-1}}^!} x_1^{i-j},$$

$$\Delta(\hat{z}_i) = \hat{z}_i \otimes 1 + \sum_{m=0}^{i} \binom{i}{m}_{q_{11}} \frac{b_i}{b_m} y_1^{i-m} g_1^{-m} g_2^{-1} \otimes \hat{z}_m \tag{2}$$

for all $i,j\in\mathbb{N}_0$ with $j\le i$. In particular,

$$z_i=0\Leftrightarrow b_i(i)^!_{q_{11}}=0\Leftrightarrow\langle\hat z_i,z_i\rangle=0. \tag{3}$$

Note that for all $i\in\mathbb{N}$ one has

$$\frac{\langle\hat z_{i+1},z_{i+1}\rangle}{\langle\hat z_i,z_i\rangle}-\frac{\langle\hat z_i,z_i\rangle}{\langle\hat z_{i-1},z_{i-1}\rangle}=\chi(z_i,x_1)^{-1}-\chi(x_1,z_i). \tag{4}$$

Similarly let $u_0:=x_1$, $u_{i+1}:=u_ix_2-q_{12}q_{22}^ix_2u_i$, and $\hat u_i:=\iota(u_i)$ for $i\in\mathbb{N}_0$. Set $c_0:=1$ and $c_i:=\prod_{j=0}^{i-1}(1-q_{12}q_{21}q_{22}^j)$ for $i\in\mathbb{N}$. Then one proves by induction over i that

$$\langle y_1,u_i\rangle=\delta_{i0},\qquad \langle y_2,u_i\rangle=q_{21}^{-1}(1-q_{12}q_{21}q_{22}^{i-1})(i)_{q_{22}^{-1}}u_{i-1}$$

for all $i\in\mathbb{N}_0$. In particular, $u_i=0$ if and only if $c_i(i)^!_{q_{22}}=0$. Because of the symmetry of the conditions for $u_i=0$ and $z_i=0$ one can choose the order of the basis vectors x_1 and x_2 in such a manner that

$$\min\{i\in\mathbb{N}\,|\,u_i=0\}\le\min\{i\in\mathbb{N}\,|\,z_i=0\} \tag{A3}$$

whenever (A0) holds. Thus assumptions (A0) and (A3) imply that either $z_1=0$ or $z_2=u_2=0$ or $z_2\ne0$, i. e.

$$q_{12}q_{21}=1\text{ or }(1-q_{11}q_{12}q_{21})(1+q_{11})=(1-q_{12}q_{21}q_{22})(1+q_{22})=0\text{ or }z_2\ne0. \tag{A4}$$

4.2 Assume that $z_i\ne0$ for some $i\in\mathbb{N}$. To shorten notation we define $p_i:=\chi(z_i,z_i)^{-1}$. By Lemma 1 and (2) one gets $\langle\hat z_i,z_i^m\rangle=(m)_{p_i}\langle\hat z_i,z_i\rangle z_i^{m-1}$ for all $m>0$. Since $z_i^m=0$ for some $m\in\mathbb{N}$ and $\langle\hat z_i,z_i\rangle\ne0$ assumption (A0) implies $p_i\ne1$.

Suppose that $z_{i+1}\ne0$ for some $i\in\mathbb{N}$. If $p_i=-1$ then

$$\langle\hat z_{i-1},z_i^2\rangle=q_{21}^{-i}b_i(i)^!_{q_{11}^{-1}}(x_1z_i+\chi(z_{i-1},z_i)^{-1}z_ix_1)=q_{21}^{-i}b_i(i)^!_{q_{11}^{-1}}z_{i+1},$$

$$\begin{aligned}\langle\hat z_{i+1}\hat z_{i-1},z_i^{2m}\rangle&=\langle\hat z_i,z_i\rangle\sum_{j=0}^{m-1}\langle\hat z_{i+1},\chi(z_{i-1},z_i)^{-2j}z_i^{2j}z_{i+1}z_i^{2(m-1-j)}\rangle\\&=\langle\hat z_i,z_i\rangle\langle\hat z_{i+1},z_{i+1}\rangle\sum_{j=0}^{m-1}\chi(z_i,z_i)^{-4j}z_i^{2(m-1)}\end{aligned}$$

since $\langle y_1, z_i\rangle = \langle y_1, z_{i+1}\rangle = \langle \hat{z}_{i+1}, z_i\rangle = 0$. As the characteristic of k is zero this yields that $z_i^{2m} \neq 0$ for all $m > 0$. Therefore (A0) implies that

$$\begin{aligned} i \in \mathbb{N}, \quad p_i^{-1}(= \chi(z_i, z_i)) = 1 \quad &\Rightarrow \quad z_i = 0, \\ i \in \mathbb{N}, \quad p_i^{-1} = -1 \quad &\Rightarrow \quad z_{i+1} = 0. \end{aligned} \tag{A5}$$

4.3 For $i \in \mathbb{N}_0$ set $z_{i,1} := z_{i+1}z_i - \chi(z_{i+1}, z_i)z_iz_{i+1}$, $\hat{z}_{i,1} := \iota(z_{i,1})$, and

$$d_{i,0} := q_{21}^{-1}(1 - q_{11}^i q_{12}q_{21})(i+1)_{q_{11}^{-1}} + \chi(z_i, z_{i+1})^{-1} - \chi(z_{i+1}, z_i).$$

Then one obtains

$$\begin{aligned} \langle \hat{z}_i, z_{i,1}\rangle = &\langle \hat{z}_i, z_{i+1}\rangle z_i - \chi(x_1, z_i)z_i\langle \hat{z}_i, z_{i+1}\rangle \\ &+ (\chi(z_i, z_{i+1})^{-1} - \chi(z_{i+1}, z_i))\langle \hat{z}_i, z_i\rangle z_{i+1} = d_{i,0}\langle \hat{z}_i, z_i\rangle z_{i+1} \end{aligned}$$

and hence $z_{i,1} = 0$ implies $d_{i,0}z_{i+1} = 0$.

Suppose now that $d_{i,0}z_{i+1} \neq 0$ for some $i \in \mathbb{N}$ and $p_i^2 + p_i + 1 = 0$. Then

$$\begin{aligned} \langle \hat{z}_{i-1}, z_i^3\rangle = &\langle \hat{z}_i, z_i\rangle(x_1z_i^2 + \chi(z_{i-1}, z_i)^{-1}z_ix_1z_i + \chi(z_{i-1}, z_i)^{-2}z_i^2x_1) \\ = &\langle \hat{z}_i, z_i\rangle(z_{i+1}z_i - \chi(x_1, z_i)\chi(z_i, z_i)z_ix_1z_i + \chi(z_{i-1}, z_i)^{-2}z_i^2x_1) \\ = &\langle \hat{z}_i, z_i\rangle(z_{i+1}z_i - \chi(z_{i+1}, z_i)z_iz_{i+1}) = \langle \hat{z}_i, z_i\rangle z_{i,1}, \end{aligned}$$

$$\begin{aligned} \langle \hat{z}_{i+1}\hat{z}_i\hat{z}_{i-1}, z_i^{3m}\rangle = &\langle \hat{z}_i, z_i\rangle \sum_{j=0}^{m-1} \chi(z_{i-1}, z_i)^{-3j}\langle \hat{z}_{i+1}\hat{z}_i, z_i^{3j}z_{i,1}z_i^{3(m-1-j)}\rangle \\ = &d_{i,0}\langle \hat{z}_i, z_i\rangle^2 \sum_{j=0}^{m-1} \chi(z_iz_{i-1}, z_i)^{-3j}\langle \hat{z}_{i+1}, z_i^{3j}z_{i+1}z_i^{3(m-1-j)}\rangle \\ = &md_{i,0}\langle \hat{z}_i, z_i\rangle^2\langle \hat{z}_{i+1}, z_{i+1}\rangle z_i^{3(m-1)} \end{aligned}$$

since $\langle y_1, z_i\rangle = \langle y_1, z_{i,1}\rangle = \langle y_1, z_{i+1}\rangle = \langle \hat{z}_i, z_i^3\rangle = \langle \hat{z}_{i+1}, z_i\rangle = 0$ and $p_i^3 = 1$. As the characteristic of k is zero this yields that $z_i^{3m} \neq 0$ for all $m > 0$. Therefore using (A5) assumption (A0) implies that

$$i \in \mathbb{N}, \quad p_i^3 = 1 \quad \Rightarrow \quad d_{i,0}z_{i+1} = 0. \tag{A6}$$

4.4 In this subsection assume only that (A5) holds. Let $i \in \mathbb{N}$. If $z_{i+1} = 0$ then set $w_i := 0$. Otherwise by Equation (3) and (A5) one can define

$$w_i := z_{i+1}z_{i-1} - \chi(z_{i+1}, z_{i-1})z_{i-1}z_{i+1} - \frac{\langle \hat{z}_{i+1}, z_{i+1}\rangle}{(2)_{p_i}\langle \hat{z}_i, z_i\rangle} z_i^2.$$

Set $\hat{w}_i := \iota(w_i)$. Our aim in this subsection is to prove the following lemma.

Lemma 6. *Assume that (A5) holds and let $i \in \mathbb{N}$. If $\langle \hat{z}_{m+1}\hat{z}_{m-1}, w_m\rangle = 0$ for $1 \le m \le i$ then $w_m = 0$ for $1 \le m \le i$. Otherwise* $\dim_k \mathcal{B}(V) = \infty$.

Therefore (A0) implies the condition

$$w_i = 0 \quad \text{for all } i \in \mathbb{N}. \tag{A7}$$

In order to prove Lemma 6 we can assume that $z_{i+1} \neq 0$ and hence $z_2 \neq 0$. One obtains $\langle y_1, w_i\rangle = 0$ and

$$\begin{aligned}
\langle \hat{z}_{i-1}, w_i\rangle = & \frac{\langle \hat{z}_{i+1}, z_{i+1}\rangle}{(2)_{q_{11}^{-1}}}(x_1^2 z_{i-1} - \chi(x_1^2, z_{i-1})z_{i-1}x_1^2) \\
& + \langle \hat{z}_{i-1}, z_{i-1}\rangle(\chi(z_{i-1}, z_{i+1})^{-1} - \chi(z_{i+1}, z_{i-1}))z_{i+1} \\
& - \frac{\langle \hat{z}_{i+1}, z_{i+1}\rangle}{(2)_{p_i}}(x_1 z_i + \chi(z_{i-1}, z_i)^{-1} z_i x_1) \\
= & \Bigg(\frac{\langle \hat{z}_{i+1}, z_{i+1}\rangle}{(2)_{q_{11}^{-1}}} + \langle \hat{z}_{i-1}, z_{i-1}\rangle \chi(z_{i-1}, z_{i+1})^{-1} \\
& - \langle \hat{z}_{i-1}, z_{i-1}\rangle \chi(z_{i+1}, z_{i-1}) - \frac{\langle \hat{z}_{i+1}, z_{i+1}\rangle}{(2)_{p_i}} \Bigg) z_{i+1} \\
= & (q_{11}p_i - 1)\left(\frac{q_{11}^{-1}\langle \hat{z}_{i+1}, z_{i+1}\rangle}{(2)_{q_{11}^{-1}}(2)_{p_i}} + q_{12}q_{21}^{-1}\langle \hat{z}_{i-1}, z_{i-1}\rangle(1 + q_{11}^{-1}p_i^{-1}) \right) z_{i+1}.
\end{aligned}$$

In particular, one gets

$$\langle y_2, w_1\rangle = \frac{q_{21}^{-2}(1 - q_{12}q_{21}q_{22})(1 + q_{22}^{-1})(1 + q_{11}q_{12}^2q_{21}^2q_{22})}{1 + q_{11}q_{12}q_{21}q_{22}} z_2,$$

$$\langle \hat{z}_i, w_i\rangle = 0, \qquad \langle \hat{w}_i, w_i\rangle = \langle \hat{z}_{i+1}\hat{z}_{i-1}, w_i\rangle.$$

Further, let A denote the subalgebra of $\mathcal{B}(V)$ generated by w_i and z_i. In what follows for $f \in \mathcal{B}(V^*)\#kG$ let $f\restriction_A$ denote the map $\langle f, \cdot\rangle : A \to \mathcal{B}(V)$.

Then $\langle y_1, A\rangle = 0$ and hence $\hat{z}_j\restriction_A$ is skew-primitive for all $j \in \mathbb{N}_0$. Since $\langle \hat{z}_{i+1}, A\rangle = 0$ one obtains

$$\begin{aligned}\Delta(\hat{w}_i)\restriction_{A\otimes A} = {} & \Delta(\hat{z}_{i+1})(\hat{z}_{i-1}\otimes 1 + g_1^{1-i}g_2^{-1}\otimes \hat{z}_{i-1})\restriction_{A\otimes A}\\ & - \frac{q_{21}^{-1}b_{i+1}(i+1)_{q_{11}^{-1}}}{(2)_{p_i}b_i}\Delta(\hat{z}_i)(\hat{z}_i\otimes 1 + g_1^{-i}g_2^{-1}\otimes \hat{z}_i)\restriction_{A\otimes A}.\end{aligned}$$

Using Equation (2), the definition of $\hat{z}_i$, and the fact that $y_1\restriction_A = \hat{z}_m\restriction_A = 0$ for $m > i$ one obtains

$$\begin{aligned}\Delta(\hat{z}_{i+1})(\hat{z}_{i-1}\otimes 1)\restriction_{A\otimes A} = {} & \hat{z}_{i+1}\hat{z}_{i-1}\restriction_A \otimes 1\\ & + (i+1)_{q_{11}}\frac{b_{i+1}}{b_i}\chi(z_i, z_{i-1})(\hat{z}_i g_1^{-i}g_2^{-1}\otimes \hat{z}_i)\restriction_{A\otimes A},\\ \Delta(\hat{z}_{i+1})(g_1^{1-i}g_2^{-1}\otimes \hat{z}_{i-1})\restriction_{A\otimes A} = {} & (g_1^{-2i}g_2^{-2}\otimes \hat{z}_{i+1}\hat{z}_{i-1})\restriction_{A\otimes A},\\ \Delta(\hat{z}_i)(\hat{z}_i\otimes 1)\restriction_{A\otimes A} = {} & (\hat{z}_i^2\otimes 1 + \chi(z_i, z_i)\hat{z}_i g_1^{-i}g_2^{-1}\otimes \hat{z}_i)\restriction_{A\otimes A},\\ \Delta(\hat{z}_i)(g_1^{-i}g_2^{-1}\otimes \hat{z}_i)\restriction_{A\otimes A} = {} & (\hat{z}_i g_1^{-i}g_2^{-1}\otimes \hat{z}_i + g_1^{-2i}g_2^{-2}\otimes \hat{z}_i^2)\restriction_{A\otimes A}.\end{aligned}$$

These equations give that

$$\Delta(\hat{w}_i)\restriction_{A\otimes A} = (\hat{w}_i\otimes 1 + g_1^{-2i}g_2^{-2}\otimes \hat{w}_i)\restriction_{A\otimes A}.$$

Thus $\hat{w}_i\restriction_A$ is a skew-primitive endomorphism of A.

Example 1. Assume that $g_2 \in G$, $q \in k\setminus\{0\}$, $g_1 = g_2^2$, and the braiding of V is given by the matrix $\begin{pmatrix} q^4 & q^2\\ q^2 & q\end{pmatrix}$. Further, suppose that (A0) holds.

1. By (A1) q has to be a root of unity and $q^4 \neq 1$. Thus $z_1 \neq 0$.
2. If $q^5 = 1$ and $q \neq 1$ then $z_2 \neq 0$ and $\chi(z_2, z_2) = 1$ which contradicts (A5). Thus $u_2 \neq 0$.
3. If $q^4 = -1$ then $u_2 \neq 0$ and $\chi(u_2, u_2) = 1$. One gets $\langle y_1y_2^2, u_2^m\rangle = m\langle y_1y_2^2, u_2\rangle u_2^{m-1}$ which is a contradiction to (A0) and $\langle y_1y_2^2, u_2\rangle \neq 0$. Thus $z_2 \neq 0$.
4. By (A5) with $i = 1$ one has $q^9 \neq -1$.
5. If $q^{13} = -1$ and $q \neq -1$ then $z_i \neq 0$ for all $i < 13$. Further, one gets $\chi(z_6, z_6) = -1$ which is a contradiction to (A5).

6. Otherwise $\langle y_2, w_1\rangle$ and x_{21} are both nonzero. Set $W := kw_1 + kx_{21}$. By Corollary 4 one obtains $\mathcal{B}(W) \lhd \mathcal{B}(V)$ and the braiding of $\mathcal{B}(W)$ with respect to the basis $\{w_1, x_{21}\}$ of W is given by the matrix $\begin{pmatrix} q^{36} & q^{18} \\ q^{18} & q^9 \end{pmatrix}$.

Thus there are two possibilities.

- There exists an infinite chain $\cdots \lhd \mathcal{B}(V_n) \lhd \mathcal{B}(V_{n-1}) \lhd \cdots \mathcal{B}(V_1) \lhd \mathcal{B}(V)$ of Nichols algebras where V_i, $i \in \mathbb{N}$, are two dimensional Yetter–Drinfel'd modules of diagonal type.

- There exists a finite chain $\mathcal{B}(V_n) \lhd \mathcal{B}(V_{n-1}) \lhd \cdots \mathcal{B}(V_1) \lhd \mathcal{B}(V_0) = \mathcal{B}(V)$ of Nichols algebras where V_i, $0 \le i \le n$, are two dimensional Yetter–Drinfel'd modules of diagonal type, and $\mathcal{B}(V_n)$ is infinite dimensional.

Therefore $\mathcal{B}(V)$ is infinite dimensional. ∎

Example 2. Assume that $g_1 \in G$, $q \in k \setminus \{0\}$, $g_2 = g_1^3$, and the braiding of V is given by the matrix $\begin{pmatrix} q & q^3 \\ q^3 & q^9 \end{pmatrix}$. Further, suppose that (A0) holds and $q^3 \neq -1$.

1. By (A1) q has to be a root of unity and $q^9 \neq 1$. Since $q^3 \neq -1$ this yields $z_1 \neq 0$.

2. If $q \in R_7$ then $u_2 \neq 0$ and $\chi(u_2, u_2) = 1$. One gets $\langle y_1 y_2^2, u_2^m\rangle = m\langle y_1 y_2^2, u_2\rangle u_2^{m-1}$ for all $m > 0$ which contradicts (A0). Hence by (3) one obtains that $z_2 \neq 0$.

3. If $q \in R_{18}$ then $z_3 \neq 0$ and $\chi(z_3, z_3) = 1$ which contradicts (A5).

4. If $q^{15} = 1$ and $q^3 \neq 1$ then $q \in R_5 \cup R_{15}$. In the first case $z_2 \neq 0$ and $\chi(z_2, z_2) = 1$ which is a contradiction to (A5). In the second case one has $z_i = 0$ if and only if $i \ge 10$. Further, $\chi(z_2, z_2)^3 = q^{75} = 1$ and $d_{2,0} = q^{-3}(1 - q^8)(1 + q^{-1} + q^{-2}) \neq 0$. This is a contradiction to (A6).

5. If $q^{22} = -1$ then $q^2 = -1$ or $q \in R_{44}$. The first case is a contradiction to (A2). In the second case one has $z_i = 0$ if and only if $i \ge 39$. Further, $\chi(z_{19}, z_{19}) = 1$ which is a contradiction to (A5).

6. Otherwise $\langle y_2, w_1\rangle$ and x_{21} are both nonzero. Set $W := kw_1 + kx_{21}$. By Corollary 4 one obtains that $\mathcal{B}(W)\lhd\mathcal{B}(V)$ and the braiding of $\mathcal{B}(W)$ is given by the matrix $\begin{pmatrix} q^{64} & q^{32} \\ q^{32} & q^{16} \end{pmatrix}$. By the first example $\dim_k \mathcal{B}(W) = \infty$.

Therefore $\dim_k \mathcal{B}(V) < \infty$ implies $q^3 = -1$. ∎

We continue with the proof of Lemma 6. Suppose that $\langle \hat{z}_{i-1}, w_i\rangle \neq 0$ for $i \in \mathbb{N}$. Then one can apply Corollary 4 with $W = kw_i + kz_i$ which gives $\mathcal{B}(W)\lhd\mathcal{B}(V)$. By Example 1 one gets $\dim_k \mathcal{B}(V) = \infty$. This proves the second part of the Lemma.

Assume that $\langle \hat{z}_{m-1}, w_m\rangle = 0$ for $1 \leq m \leq i$. We show by induction over m that $w_m = 0$ for $1 \leq m \leq i$. If $m = 1$ then $\hat{z}_0 = y_2$ and hence $w_1 = 0$. Now turn to the induction step.

We prove by induction on $-n$ that $\langle \hat{z}_n, w_{m+1}\rangle = 0$ for $0 \leq n \leq m$. Then the case $n = 0$ proves that $w_{m+1} = 0$. By assumption $\langle \hat{z}_m, w_{m+1}\rangle = 0$. If $\langle \hat{z}_{n+1}, w_{m+1}\rangle = 0$ then $\langle y_1, \langle \hat{z}_n, w_{m+1}\rangle\rangle = 0$ and hence there exists $\lambda \in k$ such that $\langle \hat{z}_n, w_{m+1}\rangle = \lambda z_{2m+2-n}$. Thus it suffices to show that $\langle \hat{z}_{2m+2-n}\hat{z}_n, w_{m+1}\rangle (= \langle y_1^{2m+2-n} y_2 \hat{z}_n, w_{m+1}\rangle) = 0$. Since $n+2 \leq 2m+2-n$ it suffices to check that $\langle y_1^{n+2} y_2 \hat{z}_n, w_{m+1}\rangle (= \langle \hat{z}_{n+2}\hat{z}_n, w_{m+1}\rangle) = 0$. Since $n < m$ one has $\hat{w}_{n+1} = 0$ and hence the latter equation follows from the hypothesis $\langle \hat{z}_{n+1}, w_{m+1}\rangle = 0$.

4.5

Lemma 7. *Assume that (A7) is satisfied. Let $i, j \in \mathbb{N}_0$ such that $z_i \neq 0$ and $j < i-1$. Then $z_i z_j - \chi(z_i, z_j) z_j z_i \in \mathrm{Lin}_k\{z_m z_{i+j-m} \,|\, j < m < i\}$.*

Proof. We use induction over $i - j$. If $i = j + 2$ then the claim follows

from $w_{j+1}=0$. To prove the induction step note that

$$\begin{aligned} z_{i+1}z_j =&(x_1z_i-\chi(x_1,z_i)z_ix_1)z_j = x_1\chi(z_i,z_j)z_jz_i\\ &+ \text{terms in } x_1\mathrm{Lin}_k\{z_mz_{i+j-m}\,|\,j<m<i\}\oplus kz_iz_{j+1}\oplus kz_iz_jx_1\\ =&\chi(z_{i+1},z_j)z_jz_{i+1} + \text{terms in } \mathrm{Lin}_k\{z_mz_{i+j+1-m}\,|\,j<m<i+1\}\\ &+ \text{terms in } \mathrm{Lin}_k\{z_mz_{i+j-m}x_1\,|\,j\le m\le i\}. \end{aligned}$$

Since $\langle y_1, z_{i+1}z_j\rangle=0$ this gives the assertion. ∎

Corollary 8. *If (A7) holds then* $z_{i,1}=0$ *if and only if* $d_{i,0}z_{i+1}=0$.

Proof. Since $\langle y_1,z_{i,1}\rangle=0$ one has $z_{i,1}=0$ if and only if $\langle \hat{z}_{j_1}\hat{z}_{j_2},z_{i,1}\rangle=0$ whenever $j_1,j_2\in\mathbb{N}_0$ and $j_1+j_2=2i+1$. By Lemma 7 this is equivalent to the fact that $\langle \hat{z}_{j_1}\hat{z}_{j_2},z_{i,1}\rangle=0$ whenever $j_1,j_2\in\mathbb{N}_0$, $j_1\le j_2+1$, and $j_1+j_2=2i+1$. Since $\langle \hat{z}_{i+1},z_{i,1}\rangle=0$ this is the same as $\langle \hat{z}_{i+1}\hat{z}_i,z_{i,1}\rangle=0$. ∎

Corollary 9. *If (A7) holds then* $z_i^2=0$ *if and only if* $(2)_{p_i}z_i=0$.

Proof. Analogous to the proof of Corollary 8. ∎

4.6 In this subsection assume that (A5)–(A7) hold. Let $i\in\mathbb{N}$. If $z_{i,1}=0$ then set $s_i:=0$. Otherwise $d_{i,0}z_{i+1}\ne 0$ by Corollary 8, $p_i^3\ne 1$ by (A6), and $p_i\ne -1$ by (A5) and since $z_{i+1}\ne 0$. Thus one can define

$$s_i := z_{i,1}z_{i-1}-\chi(z_{i,1},z_{i-1})z_{i-1}z_{i,1}-\frac{d_{i,0}\langle \hat{z}_{i+1},z_{i+1}\rangle}{(3)^!_{p_i}\langle \hat{z}_i,z_i\rangle}z_i^3.$$

For $j\in\mathbb{N}$ let $\mathcal{I}(y_1,\hat{z}_j)$ denote the left ideal of $\mathcal{B}(V^*)$ generated by y_1 and $\hat{z}_j$. Then using (2) direct computation shows that

$$\begin{aligned} \Delta(\hat{z}_{j,1}) =&\hat{z}_{j,1}\otimes 1+g(z_{j,1})^{-1}\otimes\hat{z}_{j,1}+\chi(z_j,z_{j+1})d_{j,0}\hat{z}_{j+1}g(z_j)^{-1}\otimes\hat{z}_j\\ &+\text{terms in } (\mathcal{I}(y_1,\hat{z}_{j+2})\#kG)\otimes\mathcal{B}(V^*) \end{aligned} \tag{5}$$

for all $j\in\mathbb{N}_0$. In this subsection the following lemma will be proved.

Lemma 10. *Assume that (A5)–(A7) hold and let* $i\in\mathbb{N}$. *If* $\langle \hat{z}_{i,1}\hat{z}_{i-1},s_i\rangle=0$ *then* $s_i=0$. *Otherwise* $(3)_{-p_i}=0$ *or* $\dim_k\mathcal{B}(V)=\infty$.

Let $i \in \mathbb{N}$. Without loss of generality assume that $z_{i,1} \neq 0$. By (A7) one has $w_m = 0$ for $m \in \mathbb{N}$. Further, $\langle y_1, s_i \rangle = 0$ and

$$\langle \hat{z}_i, s_i \rangle = \langle \hat{z}_i, z_{i,1} \rangle z_{i-1} - \chi(z_{i+1}, z_{i-1}) z_{i-1} \langle \hat{z}_i, z_{i,1} \rangle - d_{i,0} \langle \hat{z}_{i+1}, z_{i+1} \rangle (2)_{p_i}^{-1} z_i^2 = 0.$$

By Lemma 7 one has $s_i = 0$ if and only if $\langle \hat{z}_{j_1} \hat{z}_{j_2} \hat{z}_{j_3}, s_i \rangle = 0$ whenever $j_1, j_2, j_3 \in \mathbb{N}_0$, $j_1 + j_2 + j_3 = 3i$, and $j_1 \leq j_2 + 1 \leq j_3 + 2$. Since $\langle \hat{z}_i, s_i \rangle = 0$ this yields that $s_i = 0$ is equivalent to $\langle \hat{z}_{i+1} \hat{z}_i \hat{z}_{i-1}, s_i \rangle = 0$.

Since $\langle \hat{z}_i, s_i \rangle = 0$ and $w_i = 0$ one has $\langle \hat{z}_{i,1} \hat{z}_{i-1}, s_i \rangle = \langle \hat{z}_{i+1} \hat{z}_i \hat{z}_{i-1}, s_i \rangle = \langle \hat{z}_{i+1} \hat{z}_{i-1,1}, s_i \rangle$. Moreover, Equation $\hat{w}_i = 0$ implies that

$$\begin{aligned} y_1 \hat{z}_{i-1,1} =& (\hat{z}_{i+1} + \chi(x_1, z_i) \hat{z}_i y_1) \hat{z}_{i-1} - \chi(z_i, z_{i-1}) (\hat{z}_i + \chi(x_1, z_{i-1}) \hat{z}_{i-1} y_1) \hat{z}_i \\ =& \left(\frac{\langle \hat{z}_{i+1}, z_{i+1} \rangle}{(2)_{p_i} \langle \hat{z}_i, z_i \rangle} + \chi(x_1, z_i) - \chi(z_i, z_{i-1}) \right) \hat{z}_i^2 + \chi(x_1, z_{i-1,1}) \hat{z}_{i-1,1} y_1 \\ =& d_{i-1,0} (2)_{p_i}^{-1} \hat{z}_i^2 + \chi(x_1, z_{i-1,1}) \hat{z}_{i-1,1} y_1 \end{aligned} \tag{6}$$

by (4). Clearly one gets $\langle \hat{z}_{i-1,1}, z_{i,1} \rangle \in k x_1^2$. Further, $z_{i+1} \neq 0$ implies that $z_2 \neq 0$ and $(2)_{q_{11}^{-1}} \neq 0$, and hence Equations $\langle y_1, z_{i,1} \rangle = 0$ and (6) give

$$\langle \hat{z}_{i-1,1}, z_{i,1} \rangle = (2)_{q_{11}^{-1}}^{-1} \langle y_1 \hat{z}_{i-1,1}, z_{i,1} \rangle x_1 = \frac{d_{i-1,0} d_{i,0} \langle \hat{z}_i, z_i \rangle \langle \hat{z}_{i+1}, z_{i+1} \rangle}{(2)_{p_i} (2)_{q_{11}^{-1}}} x_1^2.$$

Using (4) and Equations $\langle \hat{z}_i, z_{i-1,1} \rangle = 0$, $\langle \hat{z}_{i+1}, z_j x_1^{i+1-j} \rangle = 0$ for $j \leq i$, one gets

$$\begin{aligned} \langle \hat{z}_{i,1} \hat{z}_{i-1}, s_i \rangle =& \left\langle \hat{z}_{i+1} \hat{z}_{i-1,1}, z_{i,1} z_{i-1} - \chi(z_{i,1}, z_{i-1}) z_{i-1} z_{i,1} - \frac{d_{i,0} \langle \hat{z}_{i+1}, z_{i+1} \rangle}{(3)^!_{p_i} \langle \hat{z}_i, z_i \rangle} z_i^3 \right\rangle \\ =& \left\langle \hat{z}_{i+1}, \langle \hat{z}_{i-1,1}, z_{i,1} \rangle z_{i-1} + d_{i-1,0} \chi(z_{i-1}, z_{i+1})^{-1} \langle \hat{z}_i, z_{i,1} \rangle \langle \hat{z}_{i-1}, z_{i-1} \rangle \right. \\ & \left. - \frac{d_{i,0} \langle \hat{z}_{i+1}, z_{i+1} \rangle}{(3)^!_{p_i} \langle \hat{z}_i, z_i \rangle} d_{i-1,0} \langle \hat{z}_i, z_i \rangle \langle \hat{z}_{i-1}, z_i^2 \rangle \right\rangle \\ =& d_{i-1,0} d_{i,0} \langle \hat{z}_i, z_i \rangle \langle \hat{z}_{i+1}, z_{i+1} \rangle^2 \left(\frac{1}{(2)_{p_i} (2)_{q_{11}^{-1}}} \right. \\ & \left. + \chi(z_{i-1}, z_{i+1})^{-1} \frac{\langle \hat{z}_{i-1}, z_{i-1} \rangle}{\langle \hat{z}_{i+1}, z_{i+1} \rangle} - \frac{1}{(3)^!_{p_i}} \right). \end{aligned}$$

To prove the second statement of the lemma it suffices to show that $\iota(s_i)\restriction_A$ is skew-primitive where A denotes the subalgebra of $\mathcal{B}(V)$ generated by z_i and s_i. Indeed, in this case Corollary 4 with $W = kz_i + ks_i$ and Example 2 imply the assertion since $p_i \neq -1$ and $\hat{z}_i\restriction_A$ is skew-primitive.

First note that

$$\begin{aligned}
s_i =& z_{i+1}z_iz_{i-1} - \chi(z_{i+1}, z_i)z_iz_{i+1}z_{i-1} - \chi(z_{i,1}, z_{i-1})z_{i-1}z_{i,1} - \frac{d_{i,0}\langle \hat{z}_{i+1}, z_{i+1}\rangle}{(3)^!_{p_i}\langle \hat{z}_i, z_i\rangle}z_i^3 \\
=& z_{i+1}z_{i-1,1} + \chi(z_i, z_{i-1})z_{i+1}z_{i-1}z_i - \chi(z_{i+1}, z_{i-1,1})z_iz_{i-1}z_{i+1} \\
&- \chi(z_{i+1}, z_i)\frac{\langle \hat{z}_{i+1}, z_{i+1}\rangle}{(2)_{p_i}\langle \hat{z}_i, z_i\rangle}z_i^3 - \chi(z_{i,1}, z_{i-1})z_{i-1}z_{i,1} - \frac{d_{i,0}\langle \hat{z}_{i+1}, z_{i+1}\rangle}{(3)^!_{p_i}\langle \hat{z}_i, z_i\rangle}z_i^3 \\
=& z_{i+1}z_{i-1,1} - \chi(z_{i+1}, z_{i-1,1})z_{i-1,1}z_{i+1} \\
&+ \left(\chi(z_i, z_{i-1}) - \chi(z_{i+1}, z_i) - \frac{d_{i,0}}{(3)_{p_i}}\right)\frac{\langle \hat{z}_{i+1}, z_{i+1}\rangle}{(2)_{p_i}\langle \hat{z}_i, z_i\rangle}z_i^3 \\
=& z_{i+1}z_{i-1,1} - \chi(z_{i+1}, z_{i-1,1})z_{i-1,1}z_{i+1} - \frac{d_{i-1,0}\langle \hat{z}_{i+1}, z_{i+1}\rangle}{(3)^!_{p_i}\langle \hat{z}_i, z_i\rangle}z_i^3.
\end{aligned}$$

Now one has $\langle \hat{z}_{i+1}, A\rangle = 0$ and $\langle \hat{z}_i, A\rangle \subset A$ and one computes

$$\begin{aligned}
\mathit{\Delta}(\hat{z}_{i+1}\hat{z}_{i-1,1})\restriction_{A\otimes A} =& \mathit{\Delta}(\hat{z}_{i+1})(\hat{z}_{i-1,1}\otimes 1 + g(z_{i-1,1})^{-1}\otimes \hat{z}_{i-1,1} \\
&+ \chi(z_{i-1}, z_i)d_{i-1,0}\hat{z}_ig(z_{i-1})^{-1}\otimes \hat{z}_{i-1})\restriction_{A\otimes A} \\
=& \Bigg(\hat{z}_{i+1}\hat{z}_{i-1,1}\otimes 1 + \chi(z_i, z_{i-1,1})(i+1)_{q_{11}}\frac{b_{i+1}}{b_i}y_1\hat{z}_{i-1,1}g(z_i)^{-1}\otimes \hat{z}_i \\
&+ g(s_i)^{-1}\otimes \hat{z}_{i+1}\hat{z}_{i-1,1} + \chi(z_i, z_i)^2d_{i-1,0}\hat{z}_ig(z_i)^{-2}\otimes \hat{z}_{i+1}\hat{z}_{i-1}\Bigg)\restriction_{A\otimes A} \\
\overset{(6)}{=}& \Bigg(\hat{z}_{i+1}\hat{z}_{i-1,1}\otimes 1 + g(s_i)^{-1}\otimes \hat{z}_{i+1}\hat{z}_{i-1,1} \\
&+ \chi(z_i, z_i)^2\frac{d_{i-1,0}\langle \hat{z}_{i+1}, z_{i+1}\rangle}{(2)_{p_i}\langle \hat{z}_i, z_i\rangle}(\hat{z}_i^2g(z_i)^{-1}\otimes \hat{z}_i + \hat{z}_ig(z_i)^{-2}\otimes \hat{z}_i^2)\Bigg)\restriction_{A\otimes A}, \\
\mathit{\Delta}(\hat{z}_i^3)\restriction_{A\otimes A} =& \mathit{\Delta}(\hat{z}_i)^3\restriction_{A\otimes A} = (\hat{z}_i^3\otimes 1 + (3)_{p_i^{-1}}\hat{z}_i^2g(z_i)^{-1}\otimes \hat{z}_i \\
&+ (3)_{p_i^{-1}}\hat{z}_ig(z_i)^{-2}\otimes \hat{z}_i^2 + g(z_i)^{-3}\otimes \hat{z}_i^3)\restriction_{A\otimes A}.
\end{aligned}$$

This proves that $\iota(s_i)\restriction_A$ is skew-primitive.

4.7 In this subsection assume that (A5) and (A7) hold. Set

$$z_{i,2} := z_{i+1}z_{i,1} - \chi(z_{i+1}, z_{i,1})z_{i,1}z_{i+1}, \quad i \in \mathbb{N}_0.$$

Using Equation (2) and $\hat{w}_{i+1} = 0$ one obtains immediately the formulas

$$\langle \hat{z}_{i+1}, z_{i,2}\rangle = 0, \qquad \langle \hat{z}_{i+2}\hat{z}_i, z_{i,2}\rangle = 0, \qquad \langle \hat{z}_{i+1}\hat{z}_i, z_{i,2}\rangle \in kz_{i+1}. \tag{7}$$

For $i \in \mathbb{N}_0$ set

$$d_{i,1} := q_{21}^{-1}(1 - q_{11}^i q_{12}q_{21})(i+1)_{q_{11}^{-1}} + (2)_{p_{i+1}}(\chi(z_i, z_{i+1})^{-1} - \chi(z_{i+1}, z_i z_{i+1})).$$

Then using the formulas for $\varDelta(\hat{z}_i)$, $\varDelta(\hat{z}_{i,1})$ and Equation (6) one obtains that

$$\begin{aligned}\varDelta(\hat{z}_{i,2}) - \hat{z}_{i,2} \otimes 1 - g(z_{i,2})^{-1} \otimes \hat{z}_{i,2} - \chi(z_{i,1}, z_{i+1})d_{i,1}\hat{z}_{i+1}g(z_{i,1})^{-1} \otimes \hat{z}_{i,1}\\ -\chi(z_i, z_{i+1})^2 d_{i,0}d_{i,1}(2)_{p_{i+1}}^{-1}\hat{z}_{i+1}^2 g(z_i)^{-1} \otimes \hat{z}_i\end{aligned} \tag{8}$$

is an element of $(\mathcal{I}(y_1, \hat{z}_{i+2}, \hat{z}_{i+2}\hat{z}_{i+1})\#kG)\otimes\mathcal{B}(V^*)$. Here $\mathcal{I}(y_1, \hat{z}_{i+2}, \hat{z}_{i+2}\hat{z}_{i+1})$ denotes the left ideal of $\mathcal{B}(V^*)$ generated by y_1, $\hat{z}_{i+2}$, and $\hat{z}_{i+2}\hat{z}_{i+1}$. The expression (8) does not make sense if $(2)_{p_{i+1}} = 0$. In this case $z_{i+2} = 0$ by (A5) and $\hat{z}_{i+1}^2 = 0$ by Corollary 9. Thus $y_1\hat{z}_{i,1} = \chi(x_1, z_{i,1})\hat{z}_{i,1}y_1$ and the formula for $\varDelta(\hat{z}_{i,2})$ takes the form (8) without the last summand.

Now we prove that for all $i \in \mathbb{N}_0$ one has

$$\langle \hat{z}_{i+1}\hat{z}_i, z_{i,2}\rangle = d_{i,0}d_{i,1}\langle \hat{z}_i, z_i\rangle\langle \hat{z}_{i+1}, z_{i+1}\rangle z_{i+1}. \tag{9}$$

Recall from Subsection 4.3 that $\langle \hat{z}_i, z_{i,1}\rangle = d_{i,0}\langle \hat{z}_i, z_i\rangle z_{i+1}$. Therefore one gets

$$\begin{aligned}\langle \hat{z}_{i+1}\hat{z}_i, z_{i,2}\rangle &\overset{(7)}{=} \langle \hat{z}_{i,1}, z_{i+1}z_{i,1} - \chi(z_{i+1}, z_{i,1})z_{i,1}z_{i+1}\rangle\\ &\overset{(5)}{=} \langle \hat{z}_{i,1}, z_{i,1}\rangle(\chi(z_{i,1}, z_{i+1})^{-1} - \chi(z_{i+1}, z_{i,1}))z_{i+1}\\ &\quad + d_{i,0}\langle \hat{z}_{i+1}, z_{i+1}\rangle\langle \hat{z}_i, z_{i,1}\rangle\\ &= d_{i,0}\langle \hat{z}_i, z_i\rangle\langle \hat{z}_{i+1}, z_{i+1}\rangle(\chi(z_{i,1}, z_{i+1})^{-1} - \chi(z_{i+1}, z_{i,1}) + d_{i,0})z_{i+1}\end{aligned}$$

which implies (9).

Lemma 11. *Assume that (A5) and (A7) hold. Let $i \in \mathbb{N}_0$ such that $z_{i+1} \neq 0$. Then one has*

$$\langle \hat{z}_i, z_{i,2} \rangle = \begin{cases} d_{i,0} d_{i,1} (2)_{p_{i+1}}^{-1} \langle \hat{z}_i, z_i \rangle z_{i+1}^2 & \text{if } p_{i+1} \neq -1, \\ 0 & \text{if } p_{i+1} = -1. \end{cases}$$

Proof. By (7) one has $\langle \hat{z}_{i+1}, z_{i,2} \rangle = 0$. Therefore Lemma 7 implies that $\langle \hat{z}_i, z_{i,2} \rangle \in \mathrm{Lin}_k \{ z_{i+1-n} z_{i+1+n} \,|\, 0 \leq n \leq i+1 \}$. Note that $z_{i+1+n} = 0$ if and only if $\langle \hat{z}_{i+1}, z_{i+1+n} \rangle = 0$. Further, $\langle \hat{z}_{i+1} \hat{z}_i, z_{i,2} \rangle \in k z_{i+1}$ by (9). This together with Equation

$$\langle \hat{z}_{i+1}, z_{i+1-n} z_{i+1+n} \rangle = \chi(z_{i+1}, z_{i+1-n})^{-1} z_{i+1-n} \langle \hat{z}_{i+1}, z_{i+1+n} \rangle, \quad n \in \mathbb{N},$$

and the fact that the latter is an element of $(k \setminus \{0\}) z_{i+1} x_1^n$, implies that $\langle \hat{z}_i, z_{i,2} \rangle \in k z_{i+1}^2$. Now apply $\langle \hat{z}_{i+1}, \cdot \rangle$ and use (9) and Corollary 9. ∎

Corollary 12. *If (A5) and (A7) hold then $z_{i,2} = 0$ if and only if $d_{i,0} d_{i,1} (2)_{p_{i+1}} z_{i+1} = 0$.*

Proof. Similar to the proof of Corollary 8. ∎

4.8 Suppose that (A5) and (A7) hold, $\chi(z_{i,1}, z_{i,1}) = -1$ for some $i \in \mathbb{N}_0$, and $z_{i,2} \neq 0$. Then one has $\langle \hat{z}_{i,1}, z_{i,2} \rangle \in k z_{i+1}$ and $\langle \hat{z}_{i,1}, z_{i,2} \rangle \neq 0$ by Corollary 12 and Equations (7) and (9). From this one gets $\langle \hat{z}_{i,2}, z_{i,2} \rangle \neq 0$. The latter implies that $z_{i,1}^{2m} \neq 0$ for all $m > 0$. Indeed, one computes $\langle \hat{z}_i, z_{i,1}^2 \rangle = d_{i,0} \langle \hat{z}_i, z_i \rangle z_{i,2}$ and $\langle \hat{z}_{i,2} \hat{z}_i, z_{i,1}^{2m} \rangle = m \langle \hat{z}_{i,2} \hat{z}_i, z_{i,1}^2 \rangle z_{i,1}^{2m-2}$.

4.9 To obtain sufficiently many subquotients we will need the following expressions. Assume again that (A5)–(A7) hold and fix $i \in \mathbb{N}$ such that $\chi(z_{i,1}, z_{i,1}) \neq -1$ (see also 4.8) and $z_{i,2} \neq 0$. Set

$$t_i := z_{i,2} z_i - \chi(z_{i,2}, z_i) z_i z_{i,2} - \frac{d_{i,1}}{1 + \chi(z_{i,1}, z_{i,1})^{-1}} z_{i,1}^2$$

and $\hat{t}_i := \iota(t_i)$. Our aim is to apply Corollary 4 with $W = kt_i + kz_{i,1}$.

One has $\langle \hat{z}_{i+1}, t_i\rangle = 0$. Let A denote the subalgebra of $\mathcal{B}(V)$ generated by the elements t_i and $z_{i,1}$. Equations (5) and (9) yield that $\hat{z}_{i,1}\restriction_A$ is skew-primitive and $\langle \hat{z}_{i,1}, t_i\rangle = 0$. Therefore $\langle \hat{z}_{i,1}, A\rangle \subset A$. One gets $\langle \hat{z}_{i,2}, t_i\rangle = 0$ as well and hence $\hat{z}_{i,2}\restriction_A = 0$ by (8).

Note that $\langle \hat{z}_{i,1}, A\rangle \subset A$ implies $\hat{z}_{i+m}\hat{z}_{i+1}\hat{z}_i\restriction_A = \hat{z}_{i+m}\hat{z}_{i,1}\restriction_A = 0$ for $m \geq 1$. Therefore using the formulas for $\Delta(\hat{z}_i)$, $\Delta(\hat{z}_{i,1})$, and $\Delta(\hat{z}_{i,2})$ one gets

$$\begin{aligned}\Delta(\hat{t}_i)\restriction_{A\otimes A} =& \Delta(\hat{z}_{i,2})(\hat{z}_i \otimes 1 + g(z_i)^{-1}\otimes \hat{z}_i)\restriction_{A\otimes A}\\ &- \frac{d_{i,1}}{1+\chi(z_{i,1},z_{i,1})^{-1}}\Delta(\hat{z}_{i,1})(\hat{z}_{i,1}\otimes 1 + g(z_{i,1})^{-1}\otimes \hat{z}_{i,1})\restriction_{A\otimes A}.\end{aligned}$$

For the summands of this expression one computes

$$\begin{gathered}\Delta(\hat{z}_{i,2})(\hat{z}_i\otimes 1)\restriction_{A\otimes A} = (\hat{z}_{i,2}\hat{z}_i\otimes 1 + \chi(z_{i,1},z_{i,1})d_{i,1}\hat{z}_{i,1}g(z_{i,1})^{-1}\otimes \hat{z}_{i,1})\restriction_{A\otimes A},\\ \Delta(\hat{z}_{i,2})(g(z_i)^{-1}\otimes \hat{z}_i)\restriction_{A\otimes A} = (g(t_i)^{-1}\otimes \hat{z}_{i,2}\hat{z}_i)\restriction_{A\otimes A},\\ \Delta(\hat{z}_{i,1})(\hat{z}_{i,1}\otimes 1)\restriction_{A\otimes A} = (\hat{z}_{i,1}^2\otimes 1 + \chi(z_{i,1},z_{i,1})\hat{z}_{i,1}g(z_{i,1})^{-1}\otimes \hat{z}_{i,1})\restriction_{A\otimes A},\\ \Delta(\hat{z}_{i,1})(g(z_{i,1})^{-1}\otimes \hat{z}_{i,1})\restriction_{A\otimes A} = (g(z_{i,1})^{-2}\otimes \hat{z}_{i,1}^2 + \hat{z}_{i,1}g(z_{i,1})^{-1}\otimes \hat{z}_{i,1})\restriction_{A\otimes A}.\end{gathered}$$

Therefore $\Delta(\hat{t}_i)\restriction_{A\otimes A} = (\hat{t}_i\otimes 1 + g(t_i)^{-1}\otimes \hat{t}_i)\restriction_{A\otimes A}$.

To apply Corollary 4 with $W = kt_i + kz_{i,1}$ one has to check the relation $\langle \hat{t}_i, t_i\rangle \neq 0$. Since $\langle \hat{z}_{i,1}, t_i\rangle = 0$ one gets $\langle \hat{t}_i, t_i\rangle = \langle \hat{z}_{i,2}\hat{z}_i, t_i\rangle$. Recall that there exist $\mu_m \in k$ such that $\langle \hat{z}_i, z_{i,m}\rangle = \mu_m z_{i+1}^m$ for $m \in \{1,2\}$. Therefore

$$\langle \hat{z}_i, t_i\rangle \in kz_{i,2} + kz_{i,1}z_{i+1} + kz_iz_{i+1}^2.$$

Since $\langle \hat{z}_{i,1}, t_i\rangle = 0$ one obtains $\langle \hat{z}_i, t_i\rangle \in kz_{i,2}$. Note that $z_{i,2} \neq 0$ implies that $(2)_{p_{i+1}} \neq 0$ by Corollary 12. Thus using the definition of t_i one gets

$$\begin{aligned}\langle \hat{z}_i, t_i\rangle =& \left(\mu_2 + (\chi(z_i,z_{i,2})^{-1} - \chi(z_{i,2},z_i))\langle \hat{z}_i, z_i\rangle - \frac{d_{i,1}\mu_1}{1+\chi(z_{i,1},z_{i,1})^{-1}}\right)z_{i,2}\\ =& \langle \hat{z}_i, z_i\rangle\left(\frac{d_{i,0}d_{i,1}}{(2)_{p_{i+1}}} - \frac{d_{i,0}d_{i,1}}{1+\chi(z_{i,1},z_{i,1})^{-1}} + \chi(z_i,z_{i,2})^{-1} - \chi(z_{i,2},z_i)\right)z_{i,2}.\end{aligned}$$

Lemma 13. *Suppose that (A5)–(A7) hold. Let $i \in \mathbb{N}$ such that $z_{i,2} \neq 0$ and either $\chi(z_{i,1}, z_{i,1}) = -1$ or $\chi(z_{i,1}, z_{i,1}) \neq -1$, $\langle \hat{z}_i, t_i \rangle \neq 0$. Then $\dim_k \mathcal{B}(V) = \infty$.*

Proof. If $\chi(z_{i,1}, z_{i,1}) = -1$ then $\dim_k \mathcal{B}(V) = \infty$ by Subsection 4.8. Otherwise $\langle \hat{z}_i, t_i \rangle \neq 0$ implies that $\langle \hat{t}_i, t_i \rangle \neq 0$ and one can apply Corollary 4 with $W = kt_i + kz_{i,1}$. Then by Example 1 one gets $\dim_k \mathcal{B}(V) = \infty$. ∎

5 The classification

The structures worked out in the previous sections are sufficient to perform the proposed classification of finite dimensional Nichols algebras. Thus in this section we generally assume that G is an abelian group, $V \in {}^{kG}_{kG}\mathcal{YD}$ is a two-dimensional Yetter–Drinfel'd module of diagonal type and $\mathcal{B}(V)$ is the corresponding Nichols algebra such that (A0) and (A3) hold. Suppose first that in (A4) we have $z_2 = 0$. Then by (A1) and (A2) $\mathcal{B}(V)$ appears in Theorem 5(1, 2.1, 2.2, 3.1). More precisely, if $q_{11}q_{12}q_{21} = 1$ and $q_{22} = -1$ then first one has to exchange the variables x_1 and x_2. Thus to prove Theorem 5 it remains to consider the case when $z_2 \neq 0$. Set $a := \min\{i \in \mathbb{N} \mid z_{i+2} = 0\}$. By (A7) w_a has to be zero. The relations $z_{a+2} = 0$, $z_{a+1} \neq 0$, and Equation (3) imply that $(1 - q_{11}^{a+1}q_{12}q_{21})(1 - q_{11}^{a+2}) = 0$. If $q_{11}^{a+1}q_{12}q_{21} = 1$ then

$$\langle \hat{z}_{a-1}, w_a \rangle = \frac{(q_{11}^{a+1}q_{22}^{-1} - 1)\langle \hat{z}_{a-1}, z_{a-1} \rangle}{1 + q_{11}^{a}q_{22}^{-1}} q_{11}^{-1}q_{21}^{-2}(1 + q_{22}^{-1})(1 + q_{11}^{-2a-1}q_{22})z_{a+1}$$

and if $q_{11}^{a+2} = 1$ then

$$\begin{aligned}\langle \hat{z}_{a-1}, w_a \rangle =& (q_{11}^{-3}(q_{12}q_{21})^{-a}q_{22}^{-1} - 1)\frac{q_{11}^{2}q_{21}^{-2}\langle \hat{z}_{a-1}, z_{a-1} \rangle}{1 + q_{11}^{-4}(q_{12}q_{21})^{-a}q_{22}^{-1}} \times \\ & (q_{11}(q_{12}q_{21})^{a+1}q_{22} + 1)(1 + q_{11}^{-6}(q_{12}q_{21})^{1-a}q_{22}^{-1})z_{a+1}.\end{aligned}$$

Therefore $w_a = 0$ allows only the following six possibilities.

- $q_{22} = q_{11}^{a+1}$, $q_{12}q_{21} = q_{11}^{-a-1}$.

- $q_{22} = -1$, $q_{12}q_{21} = q_{11}^{-a-1}$.
- $q_{22} = -q_{11}^{2a+1}$, $q_{12}q_{21} = q_{11}^{-a-1}$.
- $q_{11}^{a+2} = 1$, $q_{22} = q_{11}^{-3}(q_{12}q_{21})^{-a}$.
- $q_{11}^{a+2} = 1$, $q_{22} = -q_{11}^{-6}(q_{12}q_{21})^{-a+1}$.
- $q_{11}^{a+2} = 1$, $q_{22} = -q_{11}^{-1}(q_{12}q_{21})^{-a-1}$.

Note that by the definition of a we may also assume that

$$q_{11}^m \neq 1 \text{ for all } m \in \mathbb{N},\, m < a+2. \tag{A8}$$

In the following subsections we will analyze each case separately.

5.1 Consider the case $q_{22} = q_{11}^{a+1}$, $q_{12}q_{21} = q_{11}^{-a-1}$, $a \geq 1$. Then the braiding is of Cartan type (cf. [3, Sect. 4]) and hence it is known for many values of q_{11} that $\dim_k \mathcal{B}(V) < \infty$ if and only if $a \leq 2$ (see e. g. Theorem 4.6 in [3]). Note that by (A1) and (A8) one has $q_{11} \in \cup_{n=a+2}^{\infty} R_n$. The settings for $a = 1$ and $a = 2$ appear in Theorem 5(2.4) up to the case $a = 1$, $q_{11} \in R_3$, which is a special case of Theorem 5(2.3). We prove $\dim_k \mathcal{B}(V) = \infty$ for $a \geq 3$.

Since $\chi(z_2, z_2) = q_{11}^{4-2(a+1)+a+1} = q_{11}^{3-a}$ one gets a contradiction to (A5) if $a = 3$.

Let $a \geq 4$. Set $n := 2$, $v_1^0 := z_{a-1}$, $v_2^0 := z_a$, $w_1^0 := z_2$, $w_2^0 := x_1$, $\deg_0(x_1) := -2$, $\deg_0(x_2) := 2a+1$, and

$$\alpha_1 := g_1^{-a} g_2^{-1}.(\cdot), \qquad \alpha_2 := g_1^{a+1} g_2.(\cdot),$$
$$Y_1 := \frac{\langle \hat{z}_a, \cdot \rangle}{\langle \hat{z}_a, z_a \rangle}, \quad Y_2(\rho) := \frac{\langle \hat{z}_2, z_2 \rangle \langle \hat{z}_a, z_a \rangle}{\langle \hat{z}_{a+1}, z_{a+1} \rangle}(z_{a+1}\rho - (g_1^{a+1} g_2.\rho) z_{a+1})$$

for $\rho \in \mathcal{B}(V)$. One gets the formulas

$$\chi(z_a, v_1^0)^{-1} = q_{11}^{-2} q_{12}^{-1} = \chi(x_1, w_1^0)^{-1}, \qquad \chi(z_a, v_2^0)^{-1} = q_{11}^{-1} = \chi(x_1, w_2^0)^{-1},$$
$$\chi(z_{a+1}, v_1^0) = q_{11}^{-a-1} q_{21}^{-2} = \chi(x_2, w_1^0)^{-1}, \quad \chi(z_{a+1}, v_2^0) = q_{21}^{-1} = \chi(x_2, w_2^0)^{-1}.$$

Further, $d_{a,0} = 0$ and hence $z_{a,1} = 0$ by Corollary 8. Since $w_a = 0$ as well the smallest subspace of $\mathcal{B}(V)$ containing kv_1^0 and kv_2^0 and stable under the action of Y_1 and Y_2 is $V_1 := kz_{a-1} + kz_a + kz_a^2$. Now one can check step by step that all assumptions of Corollary 3 are fulfilled where one has to set $n_1 := 1$, $n_2 := 1$, and $\varphi_0(z_{a-1}) := z_2$, $\varphi_0(z_a) := x_1$, $\varphi_0(z_a^2) := x_1^2$. This gives a contradiction to (A0).

5.2 Assume now that $q_{12}q_{21} = q_{11}^{-a-1}$ and $q_{22} = -1$. Then $u_2 = 0$, $p_i = -q_{11}^{i(a+1-i)}$ for all i, and

$$d_{1,0} = q_{21}^{-1}(1 - q_{11}^{-a} + q_{11}^{-2a})(1 - q_{11}^{a-1}),$$
$$d_{2,0} = q_{21}^{-1}(1 - q_{11}^{a-2})(q_{11}^{a-2} + 1 - q_{11}^{1-a} - q_{11}^{-a} + q_{11}^{1-2a} + q_{11}^{3-3a}).$$

If $a = 1$ and $q_{11} \in R_4$ then $\mathcal{B}(V)$ appears in Theorem 5(2.4). Otherwise $a = 1$ and (A8) imply that $q_{11}^2 \neq 1$ and $\mathcal{B}(V)$ appears in Theorem 5(3.2, 4.1).

Further, if $a \geq 2$ then $\langle y_2, w_1 \rangle = 0$ and

$$\langle \hat{z}_1, w_2 \rangle = q_{21}^{-3} b_1 \frac{q_{11}^{-a}(1 + q_{11}^{2a-1})}{1 + q_{11}^{a-1}}(1 - q_{11}^{-a} + q_{11}^{-2a})(1 - q_{11}^{a-2}) z_3.$$

If $a = 2$ then one gets $d_{1,1} = q_{21}^{-1}(1 - q_{11}^{-2})(1 + q_{11}^3)(1 + q_{11}^{-4})$ and $(2)_{p_2} = 1 - q_{11}^2$. Thus by (A8) and Corollary 12 one has $z_{1,2} = 0$ if and only if $q_{11} \in R_6 \cup R_8 \cup R_{12}$. The case $q_{11} \in R_6$ was already considered in Subsection 5.1. If $q_{11} \in R_8 \cup R_{12}$ then $\mathcal{B}(V)$ appears in Theorem 5(4.2). Otherwise by Lemma 13 $\langle \hat{z}_1, t_1 \rangle$ has to be zero. One computes

$$\frac{\langle \hat{z}_1, t_1 \rangle}{\langle \hat{z}_1, z_1 \rangle} = \frac{q_{11}^{-10} q_{21}^{-2}(1 - q_{11}^5)(1 + q_{11}^7)(5)_{-q_{11}^2}}{1 - q_{11}^3 + q_{11}^6} z_{1,2}$$

which yields $q_{11} \in R_5 \cup R_{14} \cup R_{20}$. In these cases $\mathcal{B}(V)$ appears in Theorem 5(4.2).

Let $a \geq 3$. Then by (A8) and Equation $\langle \hat{z}_1, w_2 \rangle = 0$ one obtains that $(q_{11}^{2a-1} + 1)(q_{11}^{2a} - q_{11}^a + 1) = 0$. This means that if $a = 3$ then $q_{11} \in R_{10} \cup R_{18}$ and if $a = 4$ then $q_{11} \in R_{14} \cup R_{24}$. In these cases $\mathcal{B}(V)$ appears in Theorem 5(4.3) and Theorem 5(4.4), respectively.

Suppose now that $a \geq 5$. Then $w_3 = 0$ implies that

$$0 = \frac{\langle \hat{z}_2, w_3\rangle}{\langle \hat{z}_2, z_2\rangle} = \frac{-q_{21}^{-2}(q_{11}^{3a-5}+1)(q_{11}^{a-3}-1)}{(3)_{q_{11}^{a-2}}} \times (q_{11}^{-2}+q_{11}^{1-a}+q_{11}^{-a}-q_{11}^{2-2a}-q_{11}^{1-2a}-q_{11}^{-2a}+q_{11}^{2-3a}+q_{11}^{1-3a}+q_{11}^{4-4a})z_4.$$

For $q_{11}^{2a-1} = -1$ and $(3)_{-q_{11}^a} = 0$ one obtains

$$\frac{\langle \hat{z}_2, w_3\rangle}{\langle \hat{z}_2, z_2\rangle} = \frac{-q_{21}^{-2}(1-q_{11}^{a-4})(q_{11}^{a-3}-1)}{(3)_{q_{11}^{a-2}}} q_{11}^{-2}(5)_{q_{11}} z_4,$$

$$\frac{\langle \hat{z}_2, w_3\rangle}{\langle \hat{z}_2, z_2\rangle} = \frac{-q_{21}^{-2}(1-q_{11}^{-5})(q_{11}^{a-3}-1)}{(3)_{q_{11}^{a-2}}} (1+q_{11}^{-2})(1-q_{11}^{4-a})z_4,$$

respectively. In both cases (A8) implies that $\langle \hat{z}_2, w_3\rangle \neq 0$ which is a contradiction to (A7).

5.3 In this subsection we consider the case $q_{12}q_{21} = q_{11}^{-a-1}$, $q_{22} = -q_{11}^{2a+1}$. Note that by (A3) one has $u_i = 0$ for some $i \in \{2, 3, \dots, a+2\}$. If $u_2 = 0$ then $q_{22} = -1$ or $q_{12}q_{21}q_{22} = 1$ and hence this case is already covered by the previous subsections.

If $a = 1$ then u_3 has to be zero as well. This gives $q_{11}^2 = -1$ or $q_{11}^6 - q_{11}^3 + 1 = 0$. However $q_{11}^2 = -1$ yields $\chi(z_1, z_1) = 1$ which is a contradiction to (A5). If $q_{11} \in R_{18}$ then $\mathcal{B}(V)$ appears in Theorem 5(5.4).

For $a = 2$ one obtains

$$d_{1,0} = q_{11}q_{21}^{-1}(3)_{q_{11}^{-1}}^2(1-q_{11}^{-1}) \neq 0 \qquad \text{by (A8)},$$
$$d_{1,1} = q_{11}^{-7}q_{21}^{-1}(1-q_{11})(1+q_{11}^2)(q_{11}^8+q_{11}^7-q_{11}^5-q_{11}^4-q_{11}^3+q_{11}+1).$$

Further, $u_2 \neq 0$ implies that $1 \neq q_{12}q_{21}q_{22} = -q_{11}^2$ and (A5) and $z_3 \neq 0$ yield $p_2 \neq -1$. Therefore from Corollary 12 we obtain that $z_{1,2} = 0$ if and only if $q_{11} \in R_{30}$. In this case $\mathcal{B}(V)$ appears in Theorem 5(5.5). Otherwise Lemma 13 gives that $\langle \hat{z}_1, t_1\rangle$ has to be zero. Therefore

$$\frac{d_{1,0}d_{1,1}}{1-q_{11}^{-3}} - \frac{d_{1,0}d_{1,1}}{1+q_{11}^{-11}} - q_{21}^{-2}q_{11}^{-11} + q_{21}^{-2}q_{11}^5 = q_{11}^{-6}q_{21}^{-2}\frac{(q_{11}^2+1)(1-q_{11}^{-5})}{q_{11}^{11}+1} \times (q_{11}^8+1)(q_{11}^{12}-q_{11}^{10}-q_{11}^9+q_{11}^7+q_{11}^6+q_{11}^5-q_{11}^3-q_{11}^2+1) = 0.$$

If $q_{11}^5 = 1$ then $q_{22} = -1$. This case was considered in Subsection 5.2. Further, as mentioned above $u_2 \neq 0$ implies $q_{11}^2 \neq -1$.

Recall that $u_2 \neq 0$, $u_3 = 0$ yields $(q_{11}^7 - 1)(3)_{-q_{11}^5} = 0$ and $u_3 \neq 0$, $u_4 = 0$ implies that $(q_{11}^{12}+1)(1+q_{11}^{10}) = 0$. Further, we assumed that $q_{11} \notin R_{30}$. Thus Equations $\langle \hat{z}_1, t_1\rangle = 0$ and $u_4 = 0$ lead to a contradiction.

Let $a \geq 3$. Then

$$\langle \hat{z}_1, w_2\rangle = \frac{-q_{11}^{-1}q_{21}^{-3}(1+q_{11}^{-2})b_1}{1-q_{11}^{-1}}(1-q_{11}^{2-a})(1-q_{11}^{-a-1})z_3.$$

Thus Equation $w_2 = 0$ gives a contradiction to (A8).

5.4 In the fourth case we suppose that $q_{11}^{a+2} = 1$ and $q_{22} = q_{11}^{-3}(q_{12}q_{21})^{-a}$. By (A8) it can be assumed that $q_{11} \in R_{a+2}$. Further, the case $(q_{12}q_{21})^{a+2} = 1$ can be excluded as well. Indeed, if $(q_{12}q_{21})^{a+2} = 1$ then $q_{11}^i q_{12}q_{21} = 1$ for some $i \in \mathbb{N}_0$, $i \leq a+1$. But $z_{a+1} \neq 0$ implies that $i < a+1$ is not possible and the case $i = a+1$ was already considered in Subsections 5.1–5.3.

If $a = 1$ then $q_{11} \in R_3$, $q_{12}q_{21}q_{22} = 1$, and $(q_{12}q_{21})^3 \neq 1$, and $\mathcal{B}(V)$ appears in Theorem 5(2.3).

For $a \geq 2$ one gets

$$\frac{\langle \hat{z}_{a-2}, w_{a-1}\rangle}{\langle \hat{z}_{a-2}, z_{a-2}\rangle} = \frac{q_{11}^2 q_{21}^{-2}(q_{11}^{-5}q_{12}q_{21}-1)}{1+q_{11}^{-6}q_{12}q_{21}}(q_{11}^2+1)(q_{11}+1)(3)_{-q_{11}^{-4}q_{12}q_{21}}z_a.$$

Assume that $a = 2$. Then $q_{11}^2 = -1$, $q_{22} = q_{11}(q_{12}q_{21})^{-2}$, and $w_1 = 0$. Further, by (A3) u_4 has to be zero. If $u_2 = 0$ then $(q_{22}+1)(q_{12}q_{21}q_{22}-1) = 0$. If $q_{12}q_{21}q_{22} = 1$ then $q_{11} = q_{12}q_{21}$ which was excluded. If $q_{22} = -1$ then $q_{12}q_{21} \in R_8$, $q_{11} = (q_{12}q_{21})^{-2}$, and $\mathcal{B}(V)$ appears in Theorem 5(4.5).

If $a = 2$, $u_2 \neq 0$, and $u_3 = 0$ then $(q_{12}q_{21}q_{22}^2 - 1)(3)_{q_{22}} = 0$. Since $(q_{12}q_{21})^4 \neq 1$ there are two possibilities:

- $q_{22} \in R_{12}$, $q_{11} = q_{22}^{-3}$, $q_{12}q_{21} = q_{22}^{-2}$. Then $\chi(z_1, z_1) = q_{22}^{-4}$ and $d_{1,0} = q_{21}^{-1}(1-q_{22}^{-2})$ which is a contradiction to (A6).

- $q_{12}q_{21} \in R_{24}$, $q_{11} = (q_{12}q_{21})^{-6}$, $q_{22} = (q_{12}q_{21})^{-8}$. Then $\mathcal{B}(V)$ appears in Theorem 5(5.3).

If $a=2$, $u_3\neq 0$, and $u_4=0$ then $(q_{22}^2+1)(q_{12}q_{21}q_{22}^3-1)=0$. Again there are two possibilities:

- $q_{11}^2=q_{22}^2=-1$. Then $q_{22}=q_{11}(q_{12}q_{21})^{-2}$ implies $(q_{12}q_{21})^4=1$. This is a contradiction to the assumption at the beginning of this subsection.
- $q_{22}^{10}=-1$, $q_{12}q_{21}=q_{22}^{-3}$, $q_{11}=q_{22}^{-5}$. The change of the role of x_1 and x_2 leads to an algebra which was already considered in Subsection 5.3.

Suppose that $a\geq 3$. Then we must have $\langle \hat{z}_{a-2},w_{a-1}\rangle=0$ by (A7). If $q_{12}q_{21}=q_{11}^5$ then $(q_{12}q_{21})^{a+2}=1$ which was excluded. Further, (A8) gives $q_{11}^4\neq 1$. Thus one obtains $q_{11}^{-8}(q_{12}q_{21})^2-q_{11}^{-4}q_{12}q_{21}+1=0$ and hence $q_{11}^{-12}q_{12}^3q_{21}^3=-1$. Then Equation

$$0=\frac{\langle \hat{z}_{a-3},w_{a-2}\rangle}{\langle \hat{z}_{a-3},z_{a-3}\rangle}=\frac{q_{12}^2(q_{11}^{-12}q_{12}^2q_{21}^2-1)}{1+q_{11}^{-13}q_{12}^2q_{21}^2}(3)_{-q_{11}^{-1}}(5)_{q_{11}^{-1}}(q_{11}^{-7}q_{12}q_{21}-1)z_{a-1}$$

together with $(q_{12}q_{21})^{a+2}\neq 1$ and $q_{11}^{-12}q_{12}^2q_{21}^2=-(q_{12}q_{21})^{-1}$ implies that $(3)_{-q_{11}}(5)_{q_{11}}(q_{12}q_{21}+1)=0$.

If $(3)_{-q_{11}}=0$ then $q_{11}^3=-1$. Since $q_{11}\in R_{a+2}$ one gets $a=4$. On the other hand, $(q_{11}^{-4}q_{12}q_{21})^3=-1$ implies $(q_{12}q_{21})^3=-1$ which contradicts the assumption $(q_{12}q_{21})^{a+2}\neq 1$.

If $q_{12}q_{21}=-1$ then $(3)_{-q_{11}^{-4}q_{12}q_{21}}=0$ implies that $q_{11}\in R_3\cup R_6\cup R_{12}$. Since $a\geq 3$ this is again a contradiction to $(q_{12}q_{21})^{a+2}\neq 1$.

In the remaining case one has $a=3$, $(5)_{q_{11}}=0$, and $-q_{11}q_{12}q_{21}\in R_3$. This is equivalent to $q_{12}q_{21}\in R_{30}$, $q_{11}=(q_{12}q_{21})^{-6}$, which then yields that $q_{22}=-1$. This example appears in Theorem 5(4.8).

5.5 In this subsection assume that $q_{11}\in R_{a+2}$ and $q_{22}=-q_{11}^{-6}(q_{12}q_{21})^{1-a}$ where $a\in\mathbb{N}$. Further, as in Subsection 5.4 one can exclude the case $(q_{12}q_{21})^{a+2}=1$. One has $p_{a+1}=-q_{11}^5(q_{12}q_{21})^{-2}$,

$$\begin{aligned}
\chi(z_{a-1},z_{a-1})&=-\,q_{11}^3, & \chi(z_a,z_{a-1})&=-\,q_{12}, & \chi(z_a,z_{a+1})&=-\,q_{11}^{-4}q_{12}q_{21}^2,\\
\chi(z_{a-1},z_a)&=-\,q_{21}, & \chi(z_a,z_a)&=-\,q_{11}^{-2}q_{12}q_{21}, & \chi(z_{a+1},z_a)&=-\,q_{11}^{-4}q_{12}^2q_{21}.
\end{aligned}$$

$$
\begin{aligned}
d_{a-1,0} =&q_{21}^{-1}(3)_{q_{11}}(q_{11}^{-2}q_{12}q_{21}-1),\\
d_{a,0} =&q_{11}q_{21}^{-1}(q_{11}^{3}(q_{12}q_{21})^{-1}+1)(q_{11}^{-5}(q_{12}q_{21})^2-1),\\
d_{a,1} =&-q_{11}q_{21}^{-1}(q_{11}^{-10}(q_{12}q_{21})^4-q_{11}^{-5}(q_{12}q_{21})^2+1)(1-q_{11}^{8}(q_{12}q_{21})^{-3}),\\
\frac{\langle \hat{z}_a,t_a\rangle}{\langle \hat{z}_a,z_a\rangle} =&\frac{q_{11}^{-15}q_{12}^{-3}q_{21}^{-5}((q_{12}q_{21})^4+q_{11}^{10})}{1+q_{11}^{-5}(q_{12}q_{21})^2}(q_{12}q_{21}-q_{11}^2)((q_{12}q_{21})^5+q_{11}^{13})z_{a,2}.
\end{aligned}
$$

Suppose that $a=1$. Then $q_{11}\in R_3$ and $q_{22}=-1$. Consider the equation $d_{1,0}=0$. The case $q_{12}q_{21}=-1$ was part of the previous subsection. On the other hand, if $(q_{12}q_{21})^2=q_{11}^2$ then $\mathcal{B}(V)$ appears in Theorem 5(3.2). Otherwise $d_{1,0}\neq 0$ and $(2)_{p_2}\neq 0$. Since $q_{11}^2(q_{12}q_{21})^4-q_{11}(q_{12}q_{21})^2+1=((q_{12}q_{21})^2+1)(q_{11}^2(q_{12}q_{21})^2+1)$ there are three possibilities for $d_{1,1}=0$. First if $q_{12}^2q_{21}^2=-1$ then set $q_0:=q_{11}q_{12}q_{21}$. One gets $(3)_{-q_0^2}=0$, $q_{11}=q_0^4$, $q_{12}q_{21}=-q_0^3$, $q_{22}=-1$. This example appears in Theorem 5(3.3). Next if $(q_{12}q_{21})^2=-q_{11}$ then $q_{12}q_{21}\in R_{12}$ and $q_{22}=-1$. In this case $\mathcal{B}(V)$ appears in Theorem 5(3.4). Finally, if $q_{11}=(q_{12}q_{21})^{-3}$ then $q_{12}q_{21}\in R_9$, $q_{22}=-1$, and $\mathcal{B}(V)$ appears in Theorem 5(3.5). If nothing of these equations is true then by Corollary 12 $z_{1,2}\neq 0$ and hence $\langle \hat{z}_1,t_1\rangle$ has to vanish. Since $(q_{12}q_{21})^3\neq 1$ one has $q_{12}q_{21}\neq q_{11}^2$. Since $\chi(z_1,z_1)\neq 1$ by (A5) one obtains $q_{11}q_{12}q_{21}\neq -1$. Therefore there are two remaining cases. If $(3)_{-(q_{12}q_{21})^4}=0$ and $q_{11}=-(q_{12}q_{21})^4$ then $\mathcal{B}(V)$ appears in Theorem 5(3.6). On the other hand, if $q_{12}q_{21}\in R_{30}$ and $q_{11}=-(q_{12}q_{21})^5$ then $\mathcal{B}(V)$ appears in Theorem 5(3.7).

If $a=2$ then $q_{11}^2=-1$ and $q_{22}=(q_{12}q_{21})^{-1}$. Lemma 10 implies that $(3)_{-q_{12}q_{21}}=0$ or $\langle \hat{z}_{2,1}\hat{z}_1,s_2\rangle=0$. One computes

$$
\langle \hat{z}_{2,1}\hat{z}_1,s_2\rangle =\frac{d_{1,0}d_{2,0}\langle \hat{z}_2,z_2\rangle\langle \hat{z}_3,z_3\rangle^2q_{12}q_{21}(3)_{q_{11}^{-1}q_{12}^{-2}q_{21}^{-2}}}{(1+q_{11}^{-1})(1+q_{12}^{-1}q_{21}^{-1})(3)_{q_{12}^{-1}q_{21}^{-1}}(1-q_{11}q_{12}q_{21})}.
$$

Since $(q_{12}q_{21})^4\neq 1$ one has $d_{1,0}\neq 0$. Relations $d_{2,0}=0$ and $(q_{12}q_{21})^4\neq 1$ imply that $(q_{12}q_{21})^4=-1$, $q_{11}=(q_{12}q_{21})^2$. In this case $\mathcal{B}(V)$ appears in Theorem 5(2.5). If $(3)_{q_{11}^{-1}q_{12}^{-2}q_{21}^{-2}}=0$ then $q_{12}q_{21}\in R_{24}$, $q_{11}=(q_{12}q_{21})^6$, and $\mathcal{B}(V)$ appears in Theorem 5(2.6). On the other hand, if $(3)_{-q_{12}q_{21}}=0$ then set $q_0:=q_{11}q_{12}q_{21}$. Now one has $q_{11}=q_0^3$, $q_{12}q_{21}=-q_0^4$, $(3)_{-q_0^2}=0$,

$d_{2,0} = -q_{21}^{-1}q_0(1+q_0)^2$, $d_{2,1} = -2q_{21}^{-1}q_0(3)_{q_0}$, $(2)_{p_3} = 1+q_0$, and $\langle \hat{z}_2, t_2\rangle = \langle \hat{z}_2, z_2\rangle(1-q_0^{-1})^{-1}q_{21}^{-2}q_0(1+q_0^2)^2(1+q_0)z_{2,2}$. Then Corollary 12 and Lemma 13 give a contradiction to (A0).

Suppose now that $a \geq 3$. Then

$$\frac{\langle \hat{z}_{a-2}, w_{a-1}\rangle}{\langle \hat{z}_{a-2}, z_{a-2}\rangle} = -(q_{11}^{-2}+1)\frac{q_{11}^4q_{21}^{-2}(1-q_{11}^{-5}q_{12}q_{21})(1-q_{11}^{-2}q_{12}q_{21})}{1-q_{11}^{-1}}z_a.$$

By (A8) and $(q_{12}q_{21})^{a+2} \neq 1$ this implies that $\langle \hat{z}_{a-2}, w_{a-1}\rangle \neq 0$ which is a contradiction to (A7).

5.6 Finally we have to consider Nichols algebras where $q_{11} \in R_{a+2}$, $q_{22} = -q_{11}^{-1}(q_{12}q_{21})^{-a-1}$, and $(q_{12}q_{21})^{a+2} \neq 1$ (cf. Subsect. 5.4). One has

$$\langle y_2, w_1\rangle = \frac{(1+q_{11}^{-1}(q_{12}q_{21})^{-a})(1-q_{11}(q_{12}q_{21})^{a+1})(1-(q_{12}q_{21})^{1-a})}{q_{21}^2(1-(q_{12}q_{21})^{-a})}z_2.$$

If $a=1$ then by assumption (A3) one has $u_3 = 0$. Equation $u_2 = 0$ is equivalent to $(q_{22}+1)(1-q_{12}q_{21}q_{22}) = 0$. The cases $q_{22} = -1$ and $q_{12}q_{21}q_{22} = 1$ were considered in Subsection 5.5 and 5.4, respectively. Moreover, $u_3 = 0$, $u_2 \neq 0$ imply that $(3)_{q_{22}}(1-q_{12}q_{21}q_{22}^2) = 0$. The algebra $\mathcal{B}(V)$ with $q_{12}q_{21}q_{22}^2 = 1$ was already considered in Subsection 5.3 (exchange the generators x_1 and x_2). If $(3)_{q_{22}} = 0$ then either $q_{11} = q_{22}^{-1}$ or $q_{11} = q_{22}$. In the first case one has $q_0 := q_{11}q_{12}q_{21}$, $q_0 \in R_{12}$, $q_{11} = q_0^4$, $q_{22} = -q_0^2$. Otherwise $(3)_{-(q_{12}q_{21})^2} = 0$ and $q_{11} = q_{22} = -(q_{12}q_{21})^2$. These examples appear in Theorem 5(5.1) and Theorem 5(5.2), respectively.

If $a \geq 2$ then we must have $w_1 = w_2 = 0$ by (A7). For $(q_{12}q_{21})^a = q_{12}q_{21}$, $(q_{12}q_{21})^a = -q_{11}^{-1}$, and $(q_{12}q_{21})^a = q_{11}^{-1}(q_{12}q_{21})^{-1}$ one computes

$$\frac{\langle \hat{z}_1, w_2\rangle}{\langle \hat{z}_1, z_1\rangle} = \frac{-q_{21}^{-2}(1+q_{11}^{-2})}{1-q_{11}^{-1}}q_{11}^{-1}(q_{12}q_{21}-1)(q_{11}^3q_{12}q_{21}-1)z_3,$$

$$\frac{\langle \hat{z}_1, w_2\rangle}{\langle \hat{z}_1, z_1\rangle} = \frac{q_{21}^{-2}(q_{11}^{-3}(q_{12}q_{21})^{-1}-1)}{1+q_{11}^{-4}(q_{12}q_{21})^{-1}}q_{11}^{-1}(2)_{q_{11}^{-1}}(2)_{q_{11}^{-2}}(3)_{-q_{11}^2q_{12}q_{21}}z_3,$$

$$\frac{\langle \hat{z}_1, w_2\rangle}{\langle \hat{z}_1, z_1\rangle} = \frac{-q_{21}^{-2}(1+q_{11}^{-3}(q_{12}q_{21})^{-2})}{1+q_{11}^{-2}(q_{12}q_{21})^{-1}}q_{11}^{-2}(1-q_{11}^3q_{12}q_{21})(3)_{-q_{11}q_{12}q_{21}}z_3,$$

respectively. Let $a=2$. Then $w_1=0$, $q_{11}^2=-1$, and $(q_{12}q_{21})^4\neq 1$ imply that $q_{11}(q_{12}q_{21})^2=-1$ or $q_{11}=-(q_{12}q_{21})^3$. In the first case one has $(q_{12}q_{21})^4=-1$, $q_{11}=(q_{12}q_{21})^2$, $q_{22}=(q_{12}q_{21})^{-1}$ which was already considered in Subsection 5.5. In the second case one has $(3)_{-(q_{12}q_{21})^2}=0$, $q_{11}=(q_{12}q_{21})^{-3}$, $q_{22}=-1$. Then $\mathcal{B}(V)$ appears in Theorem 5(4.6).

Assume that $a\geq 3$. Then (A8), $(q_{12}q_{21})^{a+2}\neq 1$, and $w_1=w_2=0$ imply that either Equations $(q_{12}q_{21})^a=-q_{11}^{-1}$, $(3)_{-q_{11}^2q_{12}q_{21}}=0$ or $q_{11}(q_{12}q_{21})^{a+1}=1$, $q_{11}^3(q_{12}q_{21})^2=-1$ or $q_{11}(q_{12}q_{21})^{a+1}=1$, $(3)_{-q_{11}q_{12}q_{21}}=0$ hold. Elementary computations show that in the first case there exist precisely two solutions satisfying the assumptions at the beginning of this subsection. If $a=3$, $q_{12}q_{21}\in R_{30}$, $q_{11}=-(q_{12}q_{21})^{-3}$, and $q_{22}=(q_{12}q_{21})^{-1}$ then $\mathcal{B}(V)$ appears in Theorem 5(2.7). Otherwise $a=13$, $q_{12}q_{21}\in R_{30}$, $q_{11}=(q_{12}q_{21})^2$, and $q_{22}=(q_{12}q_{21})^{-1}$. In this case $\chi(z_9,z_9)=-(q_{12}q_{21})^5$ and $d_{9,0}=2q_{21}^{-1}/(1-q_{12}^{-1}q_{21}^{-1})\neq 0$ which is a contradiction to (A6).

In the second case, i. e. if $q_{11}(q_{12}q_{21})^{a+1}=1$ and $q_{11}^3(q_{12}q_{21})^2=-1$, (A8) and $(q_{12}q_{21})^{a+2}\neq 1$ imply that $a=3$, $q_{12}q_{21}\in R_{20}$, $q_{11}=(q_{12}q_{21})^{-4}$, and $q_{22}=-1$. Then $\mathcal{B}(V)$ appears in Theorem 5(4.7).

Finally, Equations $q_{11}(q_{12}q_{21})^{a+1}=1$, $(3)_{-q_{11}q_{12}q_{21}}=0$ have to be considered. One gets $q_{11}^3(q_{12}q_{21})^{3a+3}=1$ and $(q_{12}q_{21})^{3(a+2)}=(-1)^a$. Hence $q_{11}^3(q_{12}q_{21})^{-3}=(-1)^a$ and together with $(3)_{-q_{11}q_{12}q_{21}}=0$ one obtains $q_{11}^6=(-1)^{a+1}$. By (A8) this gives $a=10$ and hence $(3)_{-(q_{12}q_{21})^{10}}=0$. Together with $1=q_{11}^{-a-2}=(q_{12}q_{21})^{(a+1)(a+2)}=(q_{12}q_{21})^{132}$ this implies that $q_{12}q_{21}$ is a 12^{th} root of unity. The latter is a contradiction to $(q_{12}q_{21})^{a+2}\neq 1$.

Now all possible settings for the structure constants q_{ij}, $i,j\in\{1,2\}$, are investigated. Thus the proof of Theorem 5 is finished.

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